

3.5 Differentiation rules and implicit work

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

KEY WORDS product rule • quotient rule • chain rule
• implicit differentiation

Objective

Differentiate $\sin kx$, $\cos kx$, $\tan kx$, e^{kx} and $\ln x$; apply the product, quotient and chain rules (including to sec, cosec and cot); use $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$; and set up exponential models. Original teaching questions for specification items P3.4.1 to P3.4.4; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

$\frac{d}{dx} \sin kx = k \cos kx$ $\frac{d}{dx} \cos kx = -k \sin kx$ $\frac{d}{dx} \tan kx = k \sec^2 kx$

$\frac{d}{dx} e^{kx} = k e^{kx}$ $\frac{d}{dx} \ln x = \frac{1}{x}$

product $(uv)' = u'v + uv'$ quotient $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ chain $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

If x is given in terms of y , flip: $\frac{dy}{dx} = 1 \bigg/ \frac{dx}{dy}$.

Worked example – product and chain together (adapted from Edexcel IAL WMA13 June 2025 Q7). Differentiate $y = e^{-x^2} \sin 3x$.

- $u = e^{-x^2}$: chain with inner $-x^2 \Rightarrow u' = -2x e^{-x^2}$.
- $v = \sin 3x \Rightarrow v' = 3 \cos 3x$. Product rule:

$\frac{dy}{dx} = -2x e^{-x^2} \sin 3x + 3 e^{-x^2} \cos 3x = e^{-x^2} (3 \cos 3x - 2x \sin 3x)$

- Check at $x = 0$: $\frac{dy}{dx} = 3$; the curve leaves the origin like $\sin 3x$ (gradient 3) ✓.

Worked example – implicit flip. Find $\frac{dy}{dx}$ for $x = \sin 3y$.

- Differentiate the given form with respect to y :

$\frac{dx}{dy} = 3 \cos 3y \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{3 \cos 3y}$

- Check at $y = 0$: $\frac{dy}{dx} = \frac{1}{3}$, and near the origin $x \approx 3y$, so $y \approx x/3$ (gradient $\frac{1}{3}$) ✓.

2 Practice

2.1 Differentiate $y = \sin 5x$. [1]

2.2 Differentiate $y = e^{-3x}$. [1]

2.3 Differentiate $y = x^2 \cos x$ using the product rule. [3]

2.4 Differentiate $y = \frac{e^x}{x}$ using the quotient rule. [3]

2.5 Differentiate $y = \ln(4x^3 + 1)$ using the chain rule. [2]

2.6 Find $\frac{dy}{dx}$ for $x = e^{2y}$. [3]

3 Exam-style questions

3.1 (Adapted from Edexcel IAL WMA13 June 2025 Q7.) The curve C has equation $y = e^{-x^2} \sin 3x$ for $0 \leq x \leq 3$. Show, using calculus, that the x -coordinates of the stationary points of C satisfy

$$x = \frac{1}{3} \arctan \frac{3}{2x}$$

[4]

3.2 A cylindrical tank drains so that the volume of water is $V = 500e^{-0.1t}$ litres after t minutes. Find the rate at which water leaves at $t = 10$, and the time at which the rate is 5 litres per minute.

[5]

Solutions

2.1 $\frac{dy}{dx} = 5 \cos 5x$.

2.2 $\frac{dy}{dx} = -3e^{-3x}$.

2.3 $\frac{dy}{dx} = 2x \cos x - x^2 \sin x$.

2.4 $\frac{dy}{dx} = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$.

2.5 $\frac{dy}{dx} = \frac{12x^2}{4x^3+1}$ (chain: outer $\ln$, inner $4x^3+1$).

2.6 $\frac{dx}{dy} = 2e^{2y} = 2x$, so $\frac{dy}{dx} = \frac{1}{2x}$.

3.1 From the worked derivative: stationary points have

$$e^{-x^2}(3 \cos 3x - 2x \sin 3x) = 0$$

$e^{-x^2} > 0$ always, so $3 \cos 3x = 2x \sin 3x$. Dividing by $\cos 3x$ (at a stationary point $\cos 3x \neq 0$, since that would force $\sin 3x = \pm 1 \neq 0$ and then $0 = \pm 2x$ fails for $x > 0$):

$$3 = 2x \tan 3x \implies \tan 3x = \frac{3}{2x} \implies 3x = \arctan \frac{3}{2x} \implies x = \frac{1}{3} \arctan \frac{3}{2x} \quad \blacksquare$$

3.2 $\frac{dV}{dt} = -50e^{-0.1t}$; at $t = 10$: $-50e^{-1} = -18.4$ litres per minute (leaving at 18.4 L/min). Rate = 5: $50e^{-0.1t} = 5 \implies e^{-0.1t} = 0.1 \implies t = 10 \ln 10 = 23.0$ minutes (3 s.f.). Check: $V(23.0) = 500e^{-2.303} = 50.0$ litres, one tenth of the initial – matching the rate being one tenth of the initial 50 L/min ✓.

Local review guidance: correct rule with both parts differentiated; simplified derivative; exponential-rate equation solved with logs. This is authored classroom guidance.