3.3 Reciprocal trig and harmonic form

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

KEY WORDS reciprocal functions • harmonic form • identity • asymptote

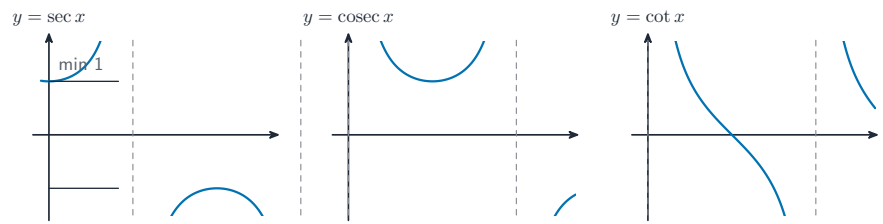
Objective

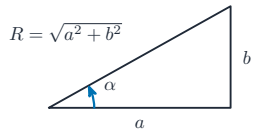
Use secant, cosecant and cotangent with their graphs and identities, the inverse trigonometric functions with restricted domains, compound and double angle formulae, and harmonic form. Original teaching questions for specification items P3.2.1 to P3.2.3; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

$\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$ $\quad$ $\sec^2\theta \equiv 1+\tan^2\theta$ $\quad$ $\operatorname{cosec}^2\theta \equiv 1+\cot^2\theta$

Inverses: $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos : [-1,1] \to [0,\pi]$, $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.
Harmonic form : $a\cos\theta + b\sin\theta = R\cos(\theta-\alpha)$, $R = \sqrt{a^2+b^2}$, $\tan\alpha = \frac{b}{a}$.





$a\cos\theta + b\sin\theta = R\cos(\theta-\alpha)$ with $\tan\alpha = \frac{b}{a}$: one amplitude R and one phase shift α replace both terms.

Sec's minimum is 1; cosec mirrors sin; cot falls through zero.

Worked example – harmonic form (adapted from Edexcel IAL WMA13 June 2025 Q2). Express $7\cos x - 24\sin x$ as $R\cos(x+\alpha)$ and hence find the maximum value of $7\cos x - 24\sin x$ and one x where it occurs.

- $R = \sqrt{49+576} = 25$; matching $25\cos(x+\alpha) = 25\cos x\cos\alpha - 25\sin x\sin\alpha$ needs $25\cos\alpha = 7$ and $25\sin\alpha = 24$, so $\tan\alpha = \frac{24}{7}$, $\alpha = 1.287$ (3 d.p.):

$$7\cos x - 24\sin x = 25\cos(x+1.287)$$

- Maximum $= R = 25$ (the amplitude), occurring when $x + 1.287 = 0$, i.e. $x = -1.287$ (plus multiples of 2π).
- Check: at $x = -1.287$: $7\cos(1.287) - 24(-\sin 1.287)$?? – direct: $\cos(-1.287 + 1.287) = \cos 0 = 1$, so the expression equals 25 ✓.

2 Practice

2.1 Write $\sec\theta$ in terms of $\cos\theta$ and state the range of $\sec\theta$. [2]

2.2 Simplify $\cot\theta\tan\theta$. [1]

2.3 State the restricted domain and range of arccos. [2]

2.4 Use $\sin 2A \equiv 2\sin A\cos A$ to write $\sin A\cos A$ as a single term. [2]

2.5 Express $12\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ with exact R and $\alpha = \arctan\frac{5}{12}$. [3]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA13 June 2025 Q2.)* Express $8\cos x - 15\sin x$ in the form $R\cos(x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$, giving R exactly and α to 3 decimal places. Hence write down the maximum value of $8\cos x - 15\sin x$ and the smallest positive x at which it occurs. [6]

3.2 *(Adapted from Edexcel IAL WMA13 June 2025 Q8a.)* Prove, using the compound angle formulae, that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

[4]

Solutions

2.1 $\sec\theta = \dfrac{1}{\cos\theta}$; since $|\cos\theta| \le 1$, $\sec\theta \ge 1$ or $\sec\theta \le -1$.

2.2 $\cot\theta\tan\theta = 1$ (where both are defined).

2.3 arccos maps $[-1, 1]$ onto $[0, \pi]$.

2.4 $\sin A\cos A = \frac{1}{2}\sin 2A$.

2.5 $R = \sqrt{144+25} = 13$, $\alpha = \arctan\frac{5}{12}$: $12\cos\theta + 5\sin\theta = 13\cos(\theta - \alpha)$. Check: $13\cos\alpha = 12$ ✓ and $13\sin\alpha = 5$ ✓ (5-12-13 triangle).

3.1 $R = \sqrt{64+225} = 17$; $17\cos\alpha = 8$, $17\sin\alpha = 15$ so $\tan\alpha = \frac{15}{8}$, $\alpha = 1.080$ (3 d.p.):

$$8\cos x - 15\sin x = 17\cos(x + 1.080)$$

Maximum $= 17$ when $x+1.080 = 0 \pmod{2\pi}$; the smallest positive x is $2\pi - 1.080 = 5.203$ (3 d.p.). Check: $\cos(5.203 + 1.080) = \cos(6.283) = \cos 2\pi = 1$ ✓.

3.2 Write $3x = 2x + x$:

$$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x\tan x}, \qquad \tan 2x = \frac{2t}{1-t^2} \quad (t = \tan x)$$

Substitute and multiply through by $(1 - t^2)$:

$$\tan 3x = \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t^2}{1-t^2}} = \frac{2t + t - t^3}{(1-t^2) - 2t^2} = \frac{3t - t^3}{1 - 3t^2} \qquad \blacksquare$$

Local review guidance: correct R and matching of both components; exact or correctly rounded values; maximum read from the amplitude with its location. This is authored classroom guidance.