3.1 Functions: rational expressions, inverses

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS domain • range • composition • inverse function • rational expression

Objective

Simplify rational expressions, work with domain, range and composition of functions, and find and verify inverse functions. Original teaching questions for specification items P3.1.1 and P3.1.2; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

- **Domain**: allowed inputs. **Range**: the outputs reached. **Composition** fg means do g first, then f.
- The **inverse** f^{-1} satisfies $f^{-1}f(x) = ff^{-1}(x) = x$; find it by writing $y = f(x)$ and solving for x.
- Simplify rational expressions by **factorising and cancelling factors** – never single terms.

Worked example – inverse with a rational function. $f(x) = \dfrac{3x}{2x-1}$, $x \neq \frac{1}{2}$. Find $f^{-1}(x)$ and state its domain.

- Write $y = \dfrac{3x}{2x-1}$ and solve for x:

$$y(2x-1) = 3x \implies 2xy - y = 3x \implies x(2y-3) = y \implies x = \frac{y}{2y-3}$$

$$f^{-1}(x) = \frac{x}{2x-3}, \qquad x \neq \tfrac{3}{2}$$

- Check: $f\left(\frac{x}{2x-3}\right) = \dfrac{3x/(2x-3)}{2x/(2x-3)-1} = \dfrac{3x}{2x-(2x-3)} = x$ ✓. The domain of f^{-1} is the range of f: as $x \to \infty$, $f(x) \to \frac{3}{2}$, so $f(x) \neq \frac{3}{2}$ ✓.

Worked example – simplify a rational expression. Simplify $\dfrac{x^2-9}{x^2+7x+12}$.

- Factorise top and bottom, then cancel the common factor:

$$\frac{(x-3)(x+3)}{(x+3)(x+4)} = \frac{x-3}{x+4} \qquad (x \neq -3, -4)$$

- Check at $x = 1$: original $\frac{-8}{20} = -\frac{2}{5}$; simplified $\frac{-2}{5}$ ✓.

2 Practice

2.1 Simplify $\dfrac{2x^2+5x-3}{x^2+2x-3}$. [3]

2.2 For $f(x) = \dfrac{x}{x+2}$ $(x \geq 0)$, state the range of f. [2]

2.3 With $f(x) = x^2$ and $g(x) = 2x+1$, find $fg(3)$ and state what you computed first. [2]

2.4 Find the inverse of $f(x) = 3x - 5$ and check it by composing. [3]

2.5 Find the inverse of $g(x) = \dfrac{x+1}{x-2}$, $x \neq 2$. [4]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA13 June 2025 Q1.)* $f(x) = \dfrac{2x}{3x+1}$, $x \geq 0$.

(a) Find the range of f.

(b) Find $f^{-1}(x)$.

(c) Solve $f^{-1}(x) = f(x)$. [6]

3.2 Express $\dfrac{5x+1}{x^2+x-2}$ as partial fractions, and hence solve $\dfrac{5x+1}{x^2+x-2} = 1$. [6]

Solutions

2.1 $\dfrac{(2x-1)(x+3)}{(x-1)(x+3)} = \dfrac{2x-1}{x-1}$ $(x \neq 1, -3)$. Check $x = 2$: original $\frac{(3)(5)}{(1)(5)} = 3$; simplified $\frac{3}{1} = 3$ ✓.

2.2 From $f(0) = 0$ increasing toward the limit 1: $0 \leq f(x) < 1$.

2.3 $g(3) = 7$ first, then $f(7) = 49$: $fg(3) = 49$. In fg, g acts first.

2.4 $y = 3x - 5 \Rightarrow x = \dfrac{y+5}{3}$, so $f^{-1}(x) = \dfrac{x+5}{3}$. Check: $f\left(\frac{x+5}{3}\right) = x + 5 - 5 = x$ ✓.

2.5 $y(x-2) = x + 1 \implies x(y-1) = 2y + 1 \implies x = \dfrac{2y+1}{y-1}$, so $g^{-1}(x) = \dfrac{2x+1}{x-1}$ $(x \neq 1)$. Check: $g\left(\frac{2x+1}{x-1}\right) = \dfrac{(2x+1)+(x-1)}{(2x+1)-2(x-1)} = \dfrac{3x}{3} = x$ ✓.

3.1 (a) $0 \leq f(x) < \frac{2}{3}$ (increasing from $f(0) = 0$ toward the horizontal limit $\frac{2}{3}$). (b) $y(3x+1) = 2x \implies x(3y-2) = -y$, so $f^{-1}(x) = \dfrac{x}{2-3x}$. (c) $\dfrac{x}{2-3x} = \dfrac{2x}{3x+1} \implies x(3x+1) = 2x(2-3x) \implies x(9x-3) = 0$, so $x = 0$ or $x = \frac{1}{3}$. Check $x = \frac{1}{3}$: $f(\frac{1}{3}) = \frac{1}{3}$ ✓.

3.2 $x^2 + x - 2 = (x+2)(x-1)$: $\dfrac{5x+1}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$ gives $5x + 1 = A(x-1) + B(x+2)$. At $x = 1$: $6 = 3B$, $B = 2$; at $x = -2$: $-9 = -3A$, $A = 3$.

$$\frac{5x+1}{x^2+x-2} = \frac{3}{x+2} + \frac{2}{x-1}$$

Setting equal to 1: $5x + 1 = x^2 + x - 2 \implies x^2 - 4x - 3 = 0 \implies x = 2 \pm \sqrt{7}$. Check $x = 2 + \sqrt{7}$: denominator $\neq 0$ ✓.

Local review guidance: correct order for composition or correct solving-for-x structure; exact inverse/range; verification by composing or by a test value. This is authored classroom guidance.