

2.2 Factor and remainder theorems

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS factor theorem • remainder theorem
• algebraic division • quotient

Objective

Use the factor theorem (including $(ax - b)$ divisors) and the remainder theorem, and divide polynomials by $(ax \pm b)$ stating quotient and remainder. Original teaching questions for specification item P2.2.1; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

For a polynomial $f(x)$:

- **Factor theorem** : $(x - b)$ is a factor $\iff f(b) = 0$. For $(ax - b)$: it is a factor $\iff f\left(\frac{b}{a}\right) = 0$.
- **Remainder theorem** : dividing by $(ax - b)$ leaves the remainder $f\left(\frac{b}{a}\right)$.
- After **algebraic division** , $f(x) = (ax - b)(\text{quotient}) + \text{remainder}$, and the quotient's degree is one less than f 's.

Worked example – factorise fully. Factorise $f(x) = x^3 + 2x^2 - 5x - 6$.

- Test small integers: $f(2) = 8 + 8 - 10 - 6 = 0$, so $(x - 2)$ is a factor.
- Divide: $x^3 + 2x^2 - 5x - 6 = (x - 2)(x^2 + 4x + 3)$.
- Factorise the quotient: $x^2 + 4x + 3 = (x + 1)(x + 3)$.

$$f(x) = (x - 2)(x + 1)(x + 3)$$

- Check: $f(0) = -6$ and $(-2)(1)(3) = -6$ ✓; roots are 2, -1, -3.

Worked example – remainder with $(2x + 1)$. Find the remainder when $2x^3 - 3x^2 + 4$ is divided by $(2x + 1)$.

- The divisor is zero at $x = -\frac{1}{2}$, so the remainder is $f\left(-\frac{1}{2}\right)$:
$$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) + 4 = -\frac{1}{4} - \frac{3}{4} + 4 = 3$$
- Check: $(2x + 1)(x^2 - 2x + 1) + 3 = 2x^3 - 4x^2 + 2x + x^2 - 2x + 1 + 3 = 2x^3 - 3x^2 + 4$ ✓.

2 Practice

2.1 Show that $(x + 1)$ is a factor of $x^3 - 7x - 6$. [2]

2.2 Hence factorise $x^3 - 7x - 6$ completely. [3]

2.3 Find the remainder when $x^3 + x^2 - 2$ is divided by $(x - 1)$, and say what your answer implies. [2]

2.4 Find the remainder when $3x^3 - 5x^2 + 7$ is divided by $(2x - 1)$. [3]

2.5 $f(x) = x^3 + ax + b$ has $(x - 2)$ as a factor and leaves remainder 15 when divided by $(x - 3)$. Find a and b . [5]

3 Exam-style questions

3.1 (Adapted from Edexcel IAL WMA12 June 2025 Q4.) $f(x) = 2x^3 + px^2 + 11x + q$, where $(x - 1)$ is a factor of $f(x)$ and $f(x)$ leaves remainder -36 when divided by $(x + 2)$. Find p and q , and verify both conditions. [5]

3.2 Using the factor theorem, fully factorise $x^3 + 4x^2 + x - 6$. [4]

Solutions

2.1 $f(-1) = -1 + 7 - 6 = 0$, so $(x + 1)$ is a factor by the factor theorem.

2.2 Divide: $x^3 - 7x - 6 = (x + 1)(x^2 - x - 6) = (x + 1)(x - 3)(x + 2)$. Check: $f(0) = -6$ and $(1)(-3)(2) = -6$ ✓.

2.3 $f(1) = 1 + 1 - 2 = 0$: the remainder is 0, so $(x - 1)$ is itself a factor.

2.4 $f(\frac{1}{2}) = 3(\frac{1}{8}) - 5(\frac{1}{4}) + 7 = \frac{3}{8} - \frac{5}{4} + 7 = \frac{49}{8}$.

2.5 $f(2) = 8 + 2a + b = 0$ and $f(3) = 27 + 3a + b = 15$. Subtracting: $19 + a = 15$, so $a = -4$; then $8 - 8 + b = 0$ gives $b = 0$. $f(x) = x^3 - 4x$. Check: $f(2) = 0$ ✓ and $f(3) = 27 - 12 = 15$ ✓.

3.1 $f(1) = 2 + p + 11 + q = 0 \implies p + q = -13$. $f(-2) = -16 + 4p - 22 + q = -36 \implies 4p + q = 2$. Subtracting: $3p = 15$, so $p = 5$ and $q = -18$. Verify: $f(1) = 2 + 5 + 11 - 18 = 0$ ✓; $f(-2) = -16 + 20 - 22 - 18 = -36$ ✓.

3.2 Test $f(-2) = -8 + 16 - 2 - 6 = 0$: $(x + 2)$ is a factor. Divide: $x^3 + 4x^2 + x - 6 = (x + 2)(x^2 + 2x - 3) = (x + 2)(x + 3)(x - 1)$. Check: $f(0) = -6$ and $(2)(3)(-1) = -6$ ✓.

Local review guidance: correct theorem applied at the right root; exact quotient and remainder; verification by rebuilding or by f evaluated. This is authored classroom guidance.