2.10 The trapezium rule

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

KEY WORDS trapezium rule • underestimate • overestimate • strip

Objective

Approximate areas with the trapezium rule from a table of values, understand how strip width and curvature control the error, and scale known integral estimates. Original teaching questions for specification item P2.8.3; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

With n strips of width h between x_0 and x_n, and heights $y_0, y_1, \ldots, y_n$:

$$\int_a^b y\,dx \approx \frac{h}{2}\big[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\big]$$

- More strips (smaller h) improve the estimate.
- A curve bending upwards (**convex**, like x^2) leaves the straight strips **above** it: **overestimate** . A curve bending downwards (**concave**, like $\sqrt{x}$) leaves them **below**: **underestimate** .

Worked example. Estimate $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ with 4 strips.

- $h = 0.5$; heights at $x = 0, 0.5, 1, 1.5, 2$: 2, 1.9365, 1.7321, 1.3229, 0.

$$\int_0^2 \sqrt{4-x^2}\,dx \approx \frac{0.5}{2}\big[2 + 0 + 2(1.9365 + 1.7321 + 1.3229)\big] = 0.25 \times 11.983 = 3.00$$

- The curve is a quarter circle of radius 2, whose true area is $\frac{1}{4}\pi(2)^2 = \pi \approx 3.14$: an **underestimate**, because the quarter circle is concave (bends downwards), so every chord lies below the arc.

2 Practice

2.1 Estimate $\displaystyle\int_0^4 x^2\,dx$ with 4 strips. [3]

2.2 Estimate $\displaystyle\int_1^3 \frac{1}{x}\,dx$ with 2 strips, and state whether your answer is an over- or underestimate. [3]

2.3 For a curve that is convex (bends upwards), state whether the trapezium rule over- or underestimates the true area, and why. [2]

2.4 Estimate $\displaystyle\int_0^1 \sqrt{x}\,dx$ with 4 strips. [3]

2.5 State two ways to improve a trapezium-rule estimate. [2]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA12 June 2025 Q5.)* The trapezium rule with five y-values and strip width $h = 1.5$ gives the estimate

$$\int_1^5 f(x)\,dx \approx 12.4$$

Using this result and making your method clear, estimate

(a) $\displaystyle\int_1^5 \frac{f(x)}{2}\,dx$

(b) $\displaystyle\int_1^5 \left(\frac{f(x)}{4} + 3x\right) dx$ [4]

3.2 *(Structure adapted from the same paper.)* Use the trapezium rule with 8 strips to estimate $\displaystyle\int_0^4 \sqrt{x}\,dx$, and compare your answer with the exact value $\frac{16}{3}$. [5]

Solutions

2.1 $h = 1$; heights $0, 1, 4, 9, 16$: $\frac{1}{2}\big[0 + 16 + 2(1 + 4 + 9)\big] = \frac{1}{2}(44) = 22$. (True value $64/3 \approx 21.3$: an overestimate, as x^2 is convex and the chords sit above.)

2.2 $h = 1$; heights at $x = 1, 2, 3$: $1, 0.5, 0.3333$: $\frac{1}{2}\big[1 + 0.3333 + 2(0.5)\big] = \frac{4}{3} = 1.333$. $y = 1/x$ is convex (bends upwards), so the chords lie above the curve: an **overestimate** (true value $\ln 3 = 1.099$).

2.3 Overestimate: a convex curve bends upwards, so every chord (strip top) sits above the curve and the strips count extra area between chord and curve.

2.4 $h = 0.25$; heights $0, 0.5, 0.7071, 0.8660, 1$: $0.125\big[0 + 1 + 2(2.0731)\big] = 0.125 \times 5.146 = 0.643$.

2.5 Use more strips (smaller h); the rule is also exact when the graph is straight, so flatter regions estimate better – in general, more strips and checking the curvature direction.

3.1 (a) Constants factor out of integrals:

$$\int_1^5 \frac{f(x)}{2}\,dx = \frac{1}{2} \times 12.4 = 6.2$$

(b) Split the integral and scale each part:

$$\int_1^5 \left(\frac{f(x)}{4} + 3x\right) dx = \frac{1}{4}(12.4) + 3\left[\frac{x^2}{2}\right]_1^5 = 3.1 + 3(12 - 0.5) = 3.1 + 34.5 = 37.6$$

Check: $\frac{5^2 - 1^2}{2} = 12$ ✓.

3.2 $h = 0.5$; heights at $x = 0, 0.5, \ldots, 4$: $0, 0.7071, 1, 1.2247, 1.4142, 1.5811, 1.7321, 1.8708, 2$.

$$\frac{0.5}{2}\big[0{+}2{+}2(0.7071{+}1{+}1.2247{+}1.4142{+}1.5811{+}1.7321{+}1.8708)\big] = 0.25\big[2{+}19.060\big] = 5.265$$

So the estimate is 5.27 (3 s.f.), below the exact $\frac{16}{3} = 5.333$: an underestimate, because $\sqrt{x}$ is concave (it rises steeply then flattens), so each chord lies below the curve and the strips undercount.

Local review guidance: correct trapezium bracket structure with $h/2$; arithmetic to the requested accuracy; error direction tied to the curvature. This is authored classroom guidance.