2.1 Proof: exhaustion and counterexample

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

KEY WORDS proof • proof by exhaustion • counterexample

Objective

Structure mathematical proofs, use proof by exhaustion over complete sets of cases, and disprove statements with a counterexample. Original teaching questions for specification items P2.1.1 to P2.1.3; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

A **proof** starts from given assumptions, moves by logical steps, and ends with the target conclusion. Quote every assumption you use; do not assume what you are proving.

- **Proof by exhaustion** lists every possible case and checks each one. The case list must be complete – "every integer is of the form $3q$, $3q+1$ or $3q+2$" is complete; "the first few integers" is not.
- **Disproof by counterexample** needs exactly one example that breaks the claim.

Worked example – exhaustion. Prove that for every integer n, the number n^2+n is even.

- The cases split on parity, and every integer is in one case:
- n even, $n=2k$: $n^2+n=4k^2+2k=2(2k^2+k)$ – even.
- n odd, $n=2k+1$: $n^2+n=(2k+1)(2k+2)$, and $2k+2$ is even, so the product is even.
- Both cases give an even number, so n^2+n is always even. ■
- Check: $n=4$: 20 ✓; $n=5$: 30 ✓; $n=-3$: 6 ✓.

Worked example – counterexample. Disprove: "2^n+1 is prime for every positive integer n."

- Try small values: $3, 5, 9, \ldots$ – and $2^3+1=9=3\times 3$ is not prime.
- One counterexample ($n=3$) disproves the claim for all n. ■

2 Practice

2.1 Disprove by counterexample: "n^2+1 is prime for all positive integers n." [2]

2.2 Prove that the sum of three consecutive integers is divisible by 3. [3]

2.3 Prove by exhaustion that n^2-1 is divisible by 3 for every positive integer n that is not a multiple of 3. [4]

2.4 Prove that the product of two odd integers is odd. [3]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA12 June 2025 Q10.)* Use algebra to prove by exhaustion that for all positive integers m that are not multiples of 3, the value of m^2+3m+2 is always a multiple of 3. [4]

3.2 Prove by exhaustion that no perfect square ends in the digit 8. [4]