

2.1 Proof: exhaustion and counterexample

Name: _____ Class: _____ Date: _____

Total: 20 marks

KEY WORDS proof • proof by exhaustion • counterexample

Objective

Structure mathematical proofs, use proof by exhaustion over complete sets of cases, and disprove statements with a counterexample. Original teaching questions for specification items P2.1.1 to P2.1.3; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

A **proof** starts from given assumptions, moves by logical steps, and ends with the target conclusion. Quote every assumption you use; do not assume what you are proving.

- **Proof by exhaustion** lists every possible case and checks each one. The case list must be complete – “every integer is of the form $3q$, $3q + 1$ or $3q + 2$ ” is complete; “the first few integers” is not.
- **Disproof by counterexample** needs exactly one example that breaks the claim.

Worked example – exhaustion. Prove that for every integer n , the number $n^2 + n$ is even.

- The cases split on parity, and every integer is in one case:
- n even, $n = 2k$: $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$ – even.
- n odd, $n = 2k + 1$: $n^2 + n = (2k + 1)(2k + 2)$, and $2k + 2$ is even, so the product is even.
- Both cases give an even number, so $n^2 + n$ is always even. ■
- Check: $n = 4$: 20 ✓; $n = 5$: 30 ✓; $n = -3$: 6 ✓.

Worked example – counterexample. Disprove: “ $2^n + 1$ is prime for every positive integer n .”

- Try small values: 3, 5, 9, ... – and $2^3 + 1 = 9 = 3 \times 3$ is not prime.
- One counterexample ($n = 3$) disproves the claim for all n . ■

2 Practice

2.1 Disprove by counterexample: " $n^2 + 1$ is prime for all positive integers n ." [2]

2.2 Prove that the sum of three consecutive integers is divisible by 3. [3]

2.3 Prove by exhaustion that $n^2 - 1$ is divisible by 3 for every positive integer n that is not a multiple of 3. [4]

2.4 Prove that the product of two odd integers is odd. [3]

3 Exam-style questions

3.1 (*Adapted from Edexcel IAL WMA12 June 2025 Q10.*) Use algebra to prove by exhaustion that for all positive integers m that are not multiples of 3, the value of $m^2 + 3m + 2$ is always a multiple of 3. [4]

3.2 Prove by exhaustion that no perfect square ends in the digit 8.

[4]
