9.3 Route inspection and the travelling salesman problem

Name: _____ Class: _____ Date: _____ **Total: 27 marks**

KEY WORDS route inspection problem • travelling salesman problem

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Objective

Solve Chinese postman problems by pairing odd nodes, and bound the classical travelling salesman problem with MST methods, shortcuts and the nearest neighbour algorithm (objectives 3.1–3.4).

1 Key knowledge and worked examples

Two route problems. Postman (every EDGE at least once, closed): add the least-weight pairing of the odd nodes; total = all edges + that pairing. TSP (every VERTEX once, complete graph, triangle inequality): upper bounds from the doubled MST improved by shortcuts, or the nearest neighbour tour; lower bounds from the MST, or delete a vertex (MST of the rest + the two cheapest edges at the deleted vertex).

Worked example (original). Edges AB 4, AC 2, BC 3, BD 5, CD 1 (total 15). Degrees A 2, B 3, C 3, D 2: odd nodes B and C; duplicate BC (3): the postman route has length 15 + 3 = 18. For the TSP on the complete network AB 4, AC 2, AD 7, BC 3, BD 5, CD 6: nearest neighbour from A gives A–C–B–D–A = 17 (upper bound); the MST is AC + BC + BD = 10, so a lower bound by deleting A is (3 + 5) + (2 + 4) = 14: the optimal tour lies in [14, 17].

2 Practice

2.1 A network has edges AB 5, BC 2, CD 4, DA 3, AC 6 with total 20. Find the odd nodes, the least duplication, and the length of the shortest closed route covering every edge. [4]

2.2 For a complete network on W, X, Y, Z with WX 3, WY 4, WZ 5, XY 2, XZ 6, YZ 7: find the nearest neighbour tour from W and its length. [3]

2.3 The MST of a complete network (triangle inequality holding) weighs 12. State the upper bound obtained by doubling the MST, and explain in one sentence why shortcuts can only improve it. [2]

3 Exam-style questions

3.1 *(Original.)* A roads inspector must travel along each road of the network with edge weights AB 6, AC 4, BC 5, CD 2, BD 7, AD 3 at least once, starting and ending at A. (a) Identify the vertices of odd degree. (b) List the possible pairings of the odd nodes with their total weights and choose the least. (c) Find the length of the inspector's shortest route. (d) State, with a reason, whether the route needs to repeat any roads. [9]

3.2 *(Original.)* A sales representative must visit each of four cities J, K, L, M once and return to the start. The complete table of shortest distances is JK 8, JL 5, JM 9, KL 4, KM 10, LM 6. (a) Use the nearest neighbour algorithm from J to find a tour and state its length. (b) Find the minimum spanning tree of the network and use it to obtain a lower bound for the optimal tour by deleting J. (c) Write down an interval containing the optimal tour length. [9]

Solutions

2.1 Degrees: A 3 (AB, AD, AC), B 2, C 3 (BC, CD, CA), D 2 – odd nodes A and C. The only pairing is AC (6): route length 20 + 6 = 26.

2.2 From W: nearest X (3); from X: nearest Y (2); from Y: nearest Z (7)? YZ = 7 vs YW = 4 but W is visited – from Y the unvisited are Z only: Y–Z (7); return Z–W (5). Tour W–X–Y–Z–W = 3 + 2 + 7 + 5 = 17.

2.3 Doubled MST: 24. A closed walk along each MST edge twice visits every vertex but repeats a vertex whenever the tree branches; replacing such repeated stretches by direct edges can only shorten the tour because the triangle inequality makes the direct edge no longer than the two-leg path.

3.1 (a) Degrees: A 3 (AB, AC, AD), B 3 (AB, BC, BD), C 3 (AC, BC, CD), D 3 (CD, BD, AD) – all four vertices are odd. (b) Pairings: AB + CD = 6 + 2 = 8; AC + BD = 4 + 7 = 11; AD + BC = 3 + 5 = 8. Least = 8 (either AB + CD or AD + BC). (c) Total of all edges = 6 + 4 + 5 + 2 + 7 + 3 = 27; shortest route = 27 + 8 = 35. (d) Yes: four odd-degree vertices cannot form a closed trail using each edge once – exactly the chosen pairing's edges (AB and CD, or AD and BC) must be repeated.

3.2 (a) From J: nearest L (5); from L: nearest K (4); from K: nearest M (10); return M–J (9). Tour J–L–K–M–J = 5 + 4 + 10 + 9 = 28. (b) MST (Kruskal): KL 4 ✓, JL 5 ✓, LM 6 ✓ (tree on J, K, L, M), total 15. Deleting J: MST of {K, L, M} = KL 4 + LM 6 = 10, plus J's two cheapest edges JL 5 + JK 8 = 13: lower bound = 10 + 13 = 23. (c) The optimal tour lies between 23 and 28 (the nearest-neighbour upper bound).

Local review guidance: odd nodes listed with degrees; ALL pairings costed before choosing; postman total = edges + chosen pairing; NN tour and the delete-a-vertex lower bound assembled from the MST of the rest plus the two cheapest at the deleted vertex. This is authored classroom guidance.