8.3 Piecewise densities and the uniform distribution

Name: _____ Class: _____ Date: _____ **Total: 30 marks**

KEY WORDS continuous uniform distribution • variance

Piecewise densities and the uniform distribution

Objective

Normalise piecewise-linear densities by exact algebra (calculator-banned style), compute moments by integration, and use the uniform distribution's facts both directions (objectives 2.2, 2.4, 3.1).

1 Key knowledge and worked examples

Two shapes. A piecewise-linear pdf normalises to (usually) a perfect square in the join point k – factorise it. Uniform on $[a, b]$: density $\frac{1}{b-a}$, mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$, $E(X^2) = \text{Var} + \text{mean}^2$; probabilities are lengths over $b - a$; an interval probability gives $b - a$, the mean gives $a + b$.

Worked example – the two shapes (WST02 June 2025 Q5, Q6). (i) $f = \frac{2x}{21}$ on $[0, k]$ then $\frac{2(6-x)}{15}$ on $(k, 6]$: normalising gives $4k^2 - 28k + 49 = 0$, i.e. $(2k - 7)^2 = 0$, so $k = 3.5$; then $E(X) = \frac{19}{6}$ and (with $E(X^2) = \frac{277}{24}$) $\text{Var} = \frac{109}{72}$. (ii) Uniform with $P(2 < X < 5) = \frac{3}{16}$ and mean 6: $b - a = 16$, $a + b = 12 \to a = -2$, $b = 14$; variance $\frac{64}{3}$, $E(X^2) = \frac{172}{3}$, $P(\frac{3}{2}X - b > a) = P(X > 8) = \frac{3}{8}$.

2 Practice

2.1 X is uniform on $[3, 11]$. Write down the mean, the variance and $P(5 < X < 9)$. [3]

2.2 X is uniform on $[a, b]$ with mean 10 and variance $\frac{64}{3}$. Find a and b. [3]

2.3 $f(x) = \frac{x}{c}$ on $[0, 4]$. Find c and $E(X)$. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WST02 June 2025 Q5.)* The continuous random variable X has probability density function $f(x) = \frac{2x}{21}$ for $0 \leq x \leq k$ and $f(x) = \frac{2(6-x)}{15}$ for $k < x \leq 6$ (0 otherwise), where k is a positive constant. Given that $E(X^2) = \frac{277}{24}$: (a) show, with all working, that $k = 3.5$; (b) use algebraic integration to find $E(X)$; (c) find $\text{Var}(X)$. [11]

3.2 *(Edexcel IAL WST02 June 2025 Q6.)* The continuous random variable X is uniformly distributed on the interval $[a, b]$. Given that $P(2 < X < 5) = \frac{3}{16}$ and $E(X) = 6$: (a) find the value of a and the value of b; (b) find the value of the constant c for which $E(cX + 1) = 3$; (c) write down the exact value of $\mathrm{Var}(X)$; (d) find the exact value of $E(X^2)$; (e) find $P(\frac{3}{2}X - b > a)$. [10]

Solutions

2.1 Mean 7; variance $\frac{64}{12} = \frac{16}{3}$; $P(5 < X < 9) = \frac{4}{8} = \frac{1}{2}$.

2.2 $\frac{(b-a)^2}{12} = \frac{64}{3}$ gives $b - a = 16$; $a + b = 20$; so $a = 2$, $b = 18$.

2.3 $\int_0^4 \frac{x}{c} = \frac{8}{c} = 1$ gives $c = 8$; $E(X) = \frac{1}{8}\int_0^4 x^2 = \frac{64}{24} = \frac{8}{3}$.

3.1 (a) Normalising: $\int_0^k \frac{2x}{21}dx + \int_k^6 \frac{2(6-x)}{15}dx = 1$, i.e. $\frac{k^2}{21} + \frac{2}{15}\left[6x - \frac{x^2}{2}\right]_k^6 = \frac{k^2}{21} + \frac{2}{15}(18 - 6k + \frac{k^2}{2}) = 1$. Multiplying by 105: $5k^2 + 14(18 - 6k + \frac{k^2}{2}) = 105$, so $5k^2 + 252 - 84k + 7k^2 = 105$: $12k^2 - 84k + 147 = 0$, i.e. $4k^2 - 28k + 49 = 0 - (2k - 7)^2 = 0$, so $k = 3.5$ ✓. (b) $E(X) = \frac{2}{21}\int_0^{3.5} x^2 dx + \frac{2}{15}\int_{3.5}^6 (6x - x^2)dx$. The first: $\frac{2}{21}\left[\frac{x^3}{3}\right]_0^{3.5} = \frac{2}{21} \cdot \frac{343}{24} = \frac{49}{36}$ (exactly $\frac{343}{252}$). The second: $\frac{2}{15}\left[3x^2 - \frac{x^3}{3}\right]_{3.5}^6$ with $3(3.5)^2 - \frac{(3.5)^3}{3} = \frac{147}{4} - \frac{343}{24} = \frac{539}{24}$, so the bracket is $36 - \frac{539}{24} = \frac{325}{24}$ and the term is $\frac{650}{360} = \frac{65}{36}$. Total: $E(X) = \frac{49+65}{36} = \frac{114}{36} = \frac{19}{6}$ ✓ (awrt 3.17). (c) $\text{Var} = \frac{277}{24} - (\frac{19}{6})^2 = \frac{277}{24} - \frac{361}{36} = \frac{831-722}{72} = \frac{109}{72}$ (awrt 1.51).

3.2 (a) $\frac{3}{b-a} = \frac{3}{16}$ gives $b - a = 16$; $\frac{a+b}{2} = 6$ gives $a + b = 12$: $2b = 28$, so $b = 14$, $a = -2$ (accept $a = 4.4$, $b = 7.6$). (b) $E(cX+1) = 6c+1 = 3$ gives $c = \frac{1}{3}$. (c) $\text{Var}(X) = \frac{16^2}{12} = \frac{64}{3}$. (d) $E(X^2) = \frac{64}{3} + 36 = \frac{172}{3}$. (e) $\frac{3}{2}X - 14 > -2$ gives $X > 8$: $P(X > 8) = \frac{14-8}{16} = \frac{3}{8}$.

Local review guidance: normalisation with both integrals over their own limits and the exact factorisation; moments by integration before any subtraction; uniform answers as exact fractions with the chosen pair of roots used consistently. This is authored classroom guidance.