4.4 Implicit differentiation and differential equations

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS implicit • connected rates of change • separable • particular solution

Objective

Differentiate implicit relations (product rule on mixed terms) and parametric curves, form differential equations from connected rates of change, and solve separable equations with boundary conditions. Original teaching questions for specification items P4.5.1, P4.5.2 and P4.6.4; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

- **Implicit** : differentiate every term with respect to x, using $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$ and the product rule on xy: $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$. Then collect the $\frac{dy}{dx}$ terms.
- **Connected rates** : $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$.
- **Separable equations** : $\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$, then apply the boundary condition for the particular solution.

Worked example – implicit tangent (adapted from Edexcel IAL WMA14 June 2025 Q1). The curve $2y^2 - 6xy + 7e^{1-2x} = 27$ passes through $\left(\frac{1}{2}, 4\right)$. Find the tangent there.

- Differentiate: $4y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} - 14e^{1-2x} = 0$, so
$$\frac{dy}{dx} = \frac{6y + 14e^{1-2x}}{4y - 6x} = \frac{24 + 14}{16 - 3} = \frac{38}{13}$$
- Tangent: $y - 4 = \frac{38}{13}(x - \frac{1}{2}) \implies 38x - 13y + 33 = 0$ (check: $19 - 52 + 33 = 0$ ✓).

Worked example – connected rates (Edexcel IAL WMA14 June 2025 Q2). With $V = \frac{4}{3}\pi r^3$ and $\frac{dV}{dt} = k$:
$$\frac{dr}{dt} = \frac{k}{4\pi r^2} \propto \frac{1}{r^2}$$
– smaller spheres melt disproportionately fast.

Worked example – separable with a condition (adapted from Edexcel IAL WMA14 June 2025 Q3). Solve $y\frac{dy}{dx} = 3\sec 2x \tan 2x$, $y = 4$ at $x = \frac{\pi}{6}$.

- Separate and integrate ($\int \sec 2x \tan 2x\,dx = \frac{1}{2}\sec 2x$):
$$\int y\,dy = \int 3\sec 2x \tan 2x\,dx \implies \frac{y^2}{2} = \frac{3}{2}\sec 2x + c \implies y^2 = 3\sec 2x + C$$
- Condition: $16 = 3\sec\frac{\pi}{3} + C = 6 + C \Rightarrow C = 10$, so
$$y^2 = 3\sec 2x + 10$$
- Check: $y = \sqrt{3(2) + 10} = 4$ at $x = \frac{\pi}{6}$ ✓.

2 Practice

2.1 Find $\frac{dy}{dx}$ for $x^2 + y^2 = 25$. [2]

2.2 Find $\frac{dy}{dx}$ for $x^2 + 2xy = 3$. [3]

2.3 A cube's edge grows at 0.1 cm/s. Find $\frac{dV}{dt}$ when the edge is 5 cm. [3]

2.4 Solve $\dfrac{dy}{dx} = \dfrac{2x}{y}$ with $y = 3$ at $x = 0$. [4]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q2.)* A sealed spherical balloon loses air at a constant 12 cm^3 per second. Show that the rate of decrease of its radius is inversely proportional to the square of the radius, and find that rate when $r = 3$ cm. [5]

3.2 *(Adapted from Edexcel IAL WMA14 June 2025 Q3.)* Solve the differential equation

$$y\frac{dy}{dx} = 6\sec 3x \tan 3x, \qquad y = 2 \text{ at } x = 0$$

giving your answer in the form $y^2 = g(x)$. [6]

Solutions

2.1 $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

2.2 $2x + 2y + 2x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x+y}{x}$.

2.3 $V = x^3$: $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt} = 3(25)(0.1) = 7.5\ \text{cm}^3/\text{s}$.

2.4 $\int y\,dy = \int 2x\,dx \Rightarrow \frac{y^2}{2} = x^2 + c$; $y(0) = 3$ gives $c = \frac{9}{2}$, so $y^2 = 2x^2 + 9$.

3.1 $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dr} = 4\pi r^2$; chain rule with $\dfrac{dV}{dt} = -12$:

$$-12 = 4\pi r^2\frac{dr}{dt} \implies \frac{dr}{dt} = -\frac{3}{\pi r^2}$$

The rate of decrease $\dfrac{3}{\pi r^2}$ is a constant multiple of $\dfrac{1}{r^2}$ – inversely proportional to r^2. ■
At $r = 3$:

$$\frac{3}{9\pi} = \frac{1}{3\pi} \approx 0.106\ \text{cm/s (3 s.f.)}$$

Check: units $\text{cm}^3/\text{s} \div \text{cm}^2 = \text{cm/s}$ ✓.

3.2 Separate: $\displaystyle\int y\,dy = \int 6\sec 3x \tan 3x\,dx$ with $\int \sec 3x \tan 3x\,dx = \frac{1}{3}\sec 3x$:

$$\frac{y^2}{2} = 2\sec 3x + c \implies y^2 = 4\sec 3x + C$$

Condition $y = 2$ at $x = 0$: $4 = 4(1) + C \Rightarrow C = 0$:

$$y^2 = 4\sec 3x$$

Check: $\sqrt{4} = 2$ at $x = 0$ ✓; differentiating back: $2y\dfrac{dy}{dx} = 12\sec 3x \tan 3x \Rightarrow y\dfrac{dy}{dx} = 6\sec 3x \tan 3x$ ✓.

Local review guidance: correct implicit/rate structure; exact derivative or particular solution; condition applied and the solution checked. This is authored classroom guidance.