3.6 Integration and numerical methods

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS recognition forms • sign change • iteration

Objective

Integrate e^{kx}, $\frac{1}{x}$ and $\sin / \cos kx$; recognise $\int \frac{f'(x)}{f(x)} dx$ and $\int f'(x)(f(x))^n dx$; use identities on squared trig functions; locate roots by sign change; and run iterations $x_{n+1} = f(x_n)$. Original teaching questions for specification items P3.5.1, P3.5.2, P3.6.1 and P3.6.2; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

$$\int e^{kx} dx = \tfrac{1}{k} e^{kx} + c \qquad \int \frac{1}{x} dx = \ln|x| + c$$

$$\int \cos kx \, dx = \tfrac{1}{k} \sin kx + c \qquad \int \sin kx \, dx = -\tfrac{1}{k} \cos kx + c$$

Recognition forms :

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n dx = \frac{(f(x))^{n+1}}{n+1} + c \quad (n \neq -1)$$

For squares: $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$, $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$, $\tan^2 A = \sec^2 A - 1$.

Numerical methods : if f is continuous and $f(a)f(b) < 0$, a root lies in (a, b); an iteration $x_{n+1} = f(x_n)$ from a start value approaches a fixed point when it converges – confirm the final accuracy with a sign change.

Worked example – identity first (adapted from Edexcel IAL WMA13 June 2025 Q5). Find $\int \sin^2 3x \, dx$.

- $\cos 6x \equiv 1 - 2\sin^2 3x$, so $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$:

$$\int \sin^2 3x \, dx = \tfrac{1}{2} \int (1 - \cos 6x) \, dx = \frac{x}{2} - \frac{\sin 6x}{12} + c$$

- Check: differentiating gives $\frac{1}{2} - \frac{1}{2}\cos 6x = \sin^2 3x$ ✓.

Worked example – recognition. Find $\displaystyle\int \frac{4x}{x^2+3}\,dx$ and $\displaystyle\int 2x\sqrt{x^2+3}\,dx$.

- First: numerator $4x$ is the derivative of x^2+3:

$$\int \frac{4x}{x^2+3}\,dx = \ln|x^2+3| + c$$

- Second: with $u = x^2+3$, $du = 2x\,dx$:

$$\int 2x\,u^{1/2}\,dx = \int u^{1/2}\,du = \tfrac{2}{3}u^{3/2} + c = \tfrac{2}{3}(x^2+3)^{3/2} + c$$

- Check the second: $\dfrac{d}{dx}\,\frac{2}{3}(x^2+3)^{3/2} = \frac{2}{3}\cdot\frac{3}{2}(x^2+3)^{1/2}\cdot 2x = 2x\sqrt{x^2+3}$ ✓.

2 Practice

2.1 Find $\displaystyle\int e^{5x}\,dx$. [2]

2.2 Find $\displaystyle\int \frac{3}{x}\,dx$ $(x > 0)$. [2]

2.3 Find $\displaystyle\int \cos 4x\,dx$. [2]

2.4 Find $\displaystyle\int \frac{2x+2}{x^2+2x+5}\,dx$. [3]

2.5 Find $\int 3x^2(x^3+1)^4\,dx$. [3]

2.6 Show that $x^3 - 3x + 1 = 0$ has a root between 0 and 1. [2]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA13 June 2025 Q5 and Q7.)*

(a) Find $\int \cos^2 2x\,dx$.

(b) Find $\int \frac{x}{x^2+4}\,dx$. [5]

3.2 *(Adapted from Edexcel IAL WMA13 June 2025 Q7.)* The equation $x = \mathrm{e}^{-x}$ has exactly one positive root.

(a) Show the root lies between 0.5 and 0.6.

(b) Use the iteration $x_{n+1} = \mathrm{e}^{-x_n}$ from $x_1 = 0.5$ to find x_3 to 3 decimal places.

(c) Verify the root is 0.567 to 3 decimal places. [7]

Solutions

2.1 $\frac{1}{5}e^{5x} + c$.

2.2 $3\ln x + c$.

2.3 $\frac{1}{4}\sin 4x + c$.

2.4 Numerator $= (x^2 + 2x + 5)'$: $\ln(x^2 + 2x + 5) + c$ (the quadratic is always positive, so no modulus needed).

2.5 With $u = x^3 + 1$, $du = 3x^2 dx$: $\frac{1}{5}(x^3 + 1)^5 + c$.

2.6 $f(x) = x^3 - 3x + 1$: $f(0) = 1 > 0$, $f(1) = -1 < 0$; f is continuous (polynomial), and the sign change over $[0, 1]$ guarantees a root in $(0, 1)$.

3.1 (a) $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$:

$$\int \cos^2 2x \, dx = \frac{x}{2} + \frac{\sin 4x}{8} + c$$

Check: $\frac{1}{2} + \frac{4}{8}\cos 4x = \frac{1}{2}(1 + \cos 4x) = \cos^2 2x$ ✓. (b) Numerator $x = \frac{1}{2}(x^2 + 4)'$: $\frac{1}{2}\ln(x^2 + 4) + c$.

3.2 (a) $g(x) = x - e^{-x}$: $g(0.5) = 0.5 - 0.6065 = -0.1065 < 0$ and $g(0.6) = 0.6 - 0.5488 = 0.0512 > 0$; g is continuous, so a root lies in $(0.5, 0.6)$. (b) $x_2 = e^{-0.5} = 0.6065$; $x_3 = e^{-0.6065} = 0.545$ (3 d.p.). (c) $g(0.5665) = 0.5665 - e^{-0.5665} = 0.5665 - 0.5675 = -0.0010 < 0$ and $g(0.5675) = 0.5675 - 0.5669 = +0.0006 > 0$: the sign change between 0.5665 and 0.5675 fixes the root as 0.567 to 3 d.p. ✓ (True root: the omega constant $0.5671\ldots$ ✓.)

Local review guidance: correct standard form or recognition of f' over f; exact integrals with $+c$; sign-change statement naming continuity, and the iteration carried with enough steps. This is authored classroom guidance.