3.5 Differentiation rules and implicit work

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

KEY WORDS product rule • quotient rule • chain rule • implicit differentiation

Objective

Differentiate $\sin kx$, $\cos kx$, $\tan kx$, e^{kx} and $\ln x$; apply the product, quotient and chain rules (including to sec, cosec and cot); use $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$; and set up exponential models. Original teaching questions for specification items P3.4.1 to P3.4.4; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

$$\frac{d}{dx}\sin kx = k\cos kx \qquad \frac{d}{dx}\cos kx = -k\sin kx \qquad \frac{d}{dx}\tan kx = k\sec^2 kx$$

$$\frac{d}{dx}e^{kx} = ke^{kx} \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\text{product } (uv)' = u'v + uv' \qquad \text{quotient } \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \text{chain } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

If x is given in terms of y, flip: $\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$.

Worked example – product and chain together (adapted from Edexcel IAL WMA13 June 2025 Q7). Differentiate $y = e^{-x^2}\sin 3x$.

- $u = e^{-x^2}$: chain with inner $-x^2 \Rightarrow u' = -2x\,e^{-x^2}$.
- $v = \sin 3x \Rightarrow v' = 3\cos 3x$. Product rule:

$$\frac{dy}{dx} = -2xe^{-x^2}\sin 3x + 3e^{-x^2}\cos 3x = e^{-x^2}\big(3\cos 3x - 2x\sin 3x\big)$$

- Check at $x = 0$: $\frac{dy}{dx} = 3$; the curve leaves the origin like $\sin 3x$ (gradient 3) ✓.

Worked example – implicit flip. Find $\dfrac{dy}{dx}$ for $x = \sin 3y$.

- Differentiate the given form with respect to y:

$$\frac{dx}{dy} = 3\cos 3y \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{3\cos 3y}$$

- Check at $y = 0$: $\frac{dy}{dx} = \frac{1}{3}$, and near the origin $x \approx 3y$, so $y \approx x/3$ (gradient $\frac{1}{3}$) ✓.

2 Practice

2.1 Differentiate $y = \sin 5x$. [1]

2.2 Differentiate $y = \mathrm{e}^{-3x}$. [1]

2.3 Differentiate $y = x^2 \cos x$ using the product rule. [3]

2.4 Differentiate $y = \dfrac{\mathrm{e}^x}{x}$ using the quotient rule. [3]

2.5 Differentiate $y = \ln(4x^3 + 1)$ using the chain rule. [2]

2.6 Find $\dfrac{dy}{dx}$ for $x = \mathrm{e}^{2y}$. [3]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA13 June 2025 Q7.)* The curve C has equation $y = \mathrm{e}^{-x^2} \sin 3x$ for $0 \leq x \leq 3$. Show, using calculus, that the x-coordinates of the stationary points of C satisfy

$$x = \tfrac{1}{3} \arctan \frac{3}{2x}$$

[4]

3.2 A cylindrical tank drains so that the volume of water is $V = 500\mathrm{e}^{-0.1t}$ litres after t minutes. Find the rate at which water leaves at $t = 10$, and the time at which the rate is 5 litres per minute. [5]

Solutions

2.1 $\dfrac{dy}{dx} = 5\cos 5x$.

2.2 $\dfrac{dy}{dx} = -3\mathrm{e}^{-3x}$.

2.3 $\dfrac{dy}{dx} = 2x\cos x - x^2\sin x$.

2.4 $\dfrac{dy}{dx} = \dfrac{\mathrm{e}^x \cdot x - \mathrm{e}^x \cdot 1}{x^2} = \dfrac{\mathrm{e}^x(x-1)}{x^2}$.

2.5 $\dfrac{dy}{dx} = \dfrac{12x^2}{4x^3+1}$ (chain: outer ln, inner $4x^3+1$).

2.6 $\dfrac{dx}{dy} = 2\mathrm{e}^{2y} = 2x$, so $\dfrac{dy}{dx} = \dfrac{1}{2x}$.

3.1 From the worked derivative: stationary points have

$$\mathrm{e}^{-x^2}\big(3\cos 3x - 2x\sin 3x\big) = 0$$

$\mathrm{e}^{-x^2} > 0$ always, so $3\cos 3x = 2x\sin 3x$. Dividing by $\cos 3x$ (at a stationary point $\cos 3x \neq 0$, since that would force $\sin 3x = \pm 1 \neq 0$ and then $0 = \pm 2x$ fails for $x > 0$):

$$3 = 2x\tan 3x \implies \tan 3x = \frac{3}{2x} \implies 3x = \arctan\frac{3}{2x} \implies x = \tfrac{1}{3}\arctan\frac{3}{2x} \qquad \blacksquare$$

3.2 $\dfrac{dV}{dt} = -50\mathrm{e}^{-0.1t}$; at $t = 10$: $-50\mathrm{e}^{-1} = -18.4$ litres per minute (leaving at 18.4 L/min). Rate = 5: $50\mathrm{e}^{-0.1t} = 5 \Rightarrow \mathrm{e}^{-0.1t} = 0.1 \Rightarrow t = 10\ln 10 = 23.0$ minutes (3 s.f.). Check: $V(23.0) = 500\mathrm{e}^{-2.303} = 50.0$ litres, one tenth of the initial – matching the rate being one tenth of the initial 50 L/min ✓.

Local review guidance: correct rule with both parts differentiated; simplified derivative; exponential-rate equation solved with logs. This is authored classroom guidance.