2.9 Definite integrals and areas

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS definite integral • area under the curve

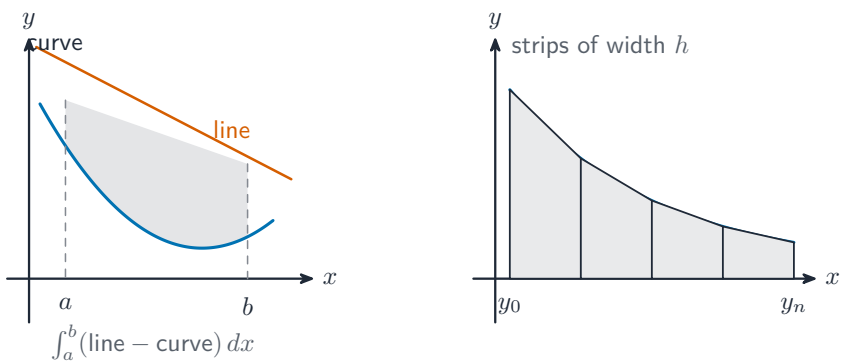
Objective

Evaluate definite integrals with the fundamental theorem, interpret them as areas, and find the area of regions bounded by a curve and a line or by two curves. Original teaching questions for specification items P2.8.1 and P2.8.2; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F'(x) = f(x)$$

For $y \geq 0$ this is the **area under the curve** . For a region between a curve and a line (or two curves), sketch first, find the intersections, and integrate **(upper − lower)**:



Integrate (line − curve) when the line is on top.

Worked example – region between curve and line. Find the area enclosed by $y = 4 - x^2$ and $y = x + 2$.

- Intersections: $4 - x^2 = x + 2 \implies x^2 + x - 2 = 0 \implies (x+2)(x-1) = 0$, so $x = -2$ and $x = 1$.

- Which is on top? At $x = 0$: curve gives 4, line gives 2 – the curve is above.

$$\text{Area} = \int_{-2}^{1} \left[(4 - x^2) - (x + 2)\right] dx = \int_{-2}^{1} (-x^2 - x + 2)\, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$

- At 1: $-\frac{1}{3} - \frac{1}{2} + 2 = \frac{7}{6}$. At -2: $\frac{8}{3} - 2 - 4 = -\frac{10}{3}$.

$$\text{Area} = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{7}{6} + \frac{20}{6} = \frac{27}{6} = 4.5$$

- Check: the region is roughly a parabolic cap about 3 wide and up to 2 tall – 4.5 is plausible ✓.

2 Practice

2.1 Evaluate $\displaystyle\int_1^3 2x\, dx$. [2]

2.2 Evaluate $\displaystyle\int_0^4 \sqrt{x}\, dx$. [3]

2.3 Evaluate $\displaystyle\int_0^{\pi/2} \cos x\, dx$. [2]

2.4 Find the area under $y = x^2$ from $x = 0$ to $x = 3$. [2]

2.5 Find the area between $y = \sqrt{x}$, $y = x$ and the lines $x = 0$, $x = 1$. [4]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA12 June 2025 Q3.)* The curve C has equation $y = 2x + \frac{3}{x} - 4$ $(x > 0)$, and the points $P(1, 1)$ and $Q(3, 3)$ lie on the line $y = x$. Find the area of the region bounded by C and the line segment PQ, giving your answer in exact form. [6]

3.2 Find the area of the region bounded by the curve $y = x^2 - 4x + 5$ and the line $y = 5$. [5]

Solutions

2.1 $\left[x^2\right]_1^3 = 9 - 1 = 8.$

2.2 $\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}(8) = \dfrac{16}{3}.$

2.3 $\left[\sin x\right]_0^{\pi/2} = 1 - 0 = 1.$

2.4 $\left[\frac{x^3}{3}\right]_0^3 = 9.$

2.5 $\sqrt{x} \geq x$ on $[0,1]$: area $= \displaystyle\int_0^1 (\sqrt{x} - x)\,dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{x^2}{2}\right]_0^1 = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}.$

3.1 First confirm P, Q lie on C: $C(1) = 2 + 3 - 4 = 1$ ✓; $C(3) = 6 + 1 - 4 = 3$ ✓. At $x = 2$ (between): $C = 4 + 1.5 - 4 = 1.5$ and the line gives 2, so the line is above:

$$\text{Area} = \int_1^3 \left[x - \left(2x + \frac{3}{x} - 4\right)\right] dx = \int_1^3 \left(-x - \frac{3}{x} + 4\right) dx$$

$$= \left[-\frac{x^2}{2} - 3\ln x + 4x\right]_1^3 = \left(-\frac{9}{2} - 3\ln 3 + 12\right) - \left(-\frac{1}{2} + 4\right) = \frac{15}{2} - 3\ln 3 - \frac{7}{2}$$

$$= 4 - 3\ln 3$$

Check: $3\ln 3 \approx 3.30$, so the area is $4 - 3.30 = 0.70$ – positive ✓ (integrating the right way round).

3.2 Intersections: $x^2 - 4x + 5 = 5 \implies x(x-4) = 0 \implies x = 0, 4$; the line is on top (at $x = 2$: curve 1, line 5).

$$\text{Area} = \int_0^4 \left[5 - (x^2 - 4x + 5)\right] dx = \int_0^4 (-x^2 + 4x)\,dx = \left[-\frac{x^3}{3} + 2x^2\right]_0^4 = -\frac{64}{3} + 32 = \frac{32}{3}$$

Check: $32/3 \approx 10.7$ – a cap 4 wide and up to 4 deep averages about 2.7 ✓.

Local review guidance: correct antiderivative and limits from the intersections; exact value; integrand ordered upper-minus-lower from a sketch check. This is authored classroom guidance.