2.2 Factor and remainder theorems

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS factor theorem • remainder theorem • algebraic division • quotient

Objective

Use the factor theorem (including $(ax-b)$ divisors) and the remainder theorem, and divide polynomials by $(ax \pm b)$ stating quotient and remainder. Original teaching questions for specification item P2.2.1; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

For a polynomial $f(x)$:

- **Factor theorem** : $(x-b)$ is a factor $\iff f(b)=0$. For $(ax-b)$: it is a factor $\iff f\left(\dfrac{b}{a}\right)=0$.
- **Remainder theorem** : dividing by $(ax-b)$ leaves the remainder $f\left(\dfrac{b}{a}\right)$.
- After **algebraic division** , $f(x)=(ax-b)(\text{quotient})+\text{remainder}$, and the quotient's degree is one less than f's.

Worked example – factorise fully. Factorise $f(x)=x^3+2x^2-5x-6$.

- Test small integers: $f(2)=8+8-10-6=0$, so $(x-2)$ is a factor.
- Divide: $x^3+2x^2-5x-6=(x-2)(x^2+4x+3)$.
- Factorise the quotient: $x^2+4x+3=(x+1)(x+3)$.

$$f(x)=(x-2)(x+1)(x+3)$$

- Check: $f(0)=-6$ and $(-2)(1)(3)=-6$ ✓; roots are $2,-1,-3$.

Worked example – remainder with $(2x+1)$. Find the remainder when $2x^3-3x^2+4$ is divided by $(2x+1)$.

- The divisor is zero at $x=-\frac{1}{2}$, so the remainder is $f\left(-\frac{1}{2}\right)$:

$$f\left(-\tfrac{1}{2}\right)=2\left(-\tfrac{1}{8}\right)-3\left(\tfrac{1}{4}\right)+4=-\tfrac{1}{4}-\tfrac{3}{4}+4=3$$

- Check: $(2x+1)(x^2-2x+1)+3=2x^3-4x^2+2x+x^2-2x+1+3=2x^3-3x^2+4$ ✓.

2 Practice

2.1 Show that $(x+1)$ is a factor of $x^3 - 7x - 6$. [2]

2.2 Hence factorise $x^3 - 7x - 6$ completely. [3]

2.3 Find the remainder when $x^3 + x^2 - 2$ is divided by $(x-1)$, and say what your answer implies. [2]

2.4 Find the remainder when $3x^3 - 5x^2 + 7$ is divided by $(2x-1)$. [3]

2.5 $f(x) = x^3 + ax + b$ has $(x-2)$ as a factor and leaves remainder 15 when divided by $(x-3)$. Find a and b. [5]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA12 June 2025 Q4.)* $f(x) = 2x^3 + px^2 + 11x + q$, where $(x-1)$ is a factor of $f(x)$ and $f(x)$ leaves remainder -36 when divided by $(x+2)$. Find p and q, and verify both conditions. [5]

3.2 Using the factor theorem, fully factorise $x^3 + 4x^2 + x - 6$. [4]

Solutions

2.1 $f(-1) = -1 + 7 - 6 = 0$, so $(x + 1)$ is a factor by the factor theorem.

2.2 Divide: $x^3 - 7x - 6 = (x+1)(x^2 - x - 6) = (x+1)(x-3)(x+2)$. Check: $f(0) = -6$ and $(1)(-3)(2) = -6$ ✓.

2.3 $f(1) = 1 + 1 - 2 = 0$: the remainder is 0, so $(x - 1)$ is itself a factor.

2.4 $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{8}\right) - 5\left(\frac{1}{4}\right) + 7 = \frac{3}{8} - \frac{5}{4} + 7 = \frac{49}{8}$.

2.5 $f(2) = 8 + 2a + b = 0$ and $f(3) = 27 + 3a + b = 15$. Subtracting: $19 + a = 15$, so $a = -4$; then $8 - 8 + b = 0$ gives $b = 0$. $f(x) = x^3 - 4x$. Check: $f(2) = 0$ ✓ and $f(3) = 27 - 12 = 15$ ✓.

3.1 $f(1) = 2 + p + 11 + q = 0 \implies p + q = -13$. $f(-2) = -16 + 4p - 22 + q = -36 \implies 4p + q = 2$. Subtracting: $3p = 15$, so $p = 5$ and $q = -18$. Verify: $f(1) = 2+5+11-18 = 0$ ✓; $f(-2) = -16 + 20 - 22 - 18 = -36$ ✓.

3.2 Test $f(-2) = -8 + 16 - 2 - 6 = 0$: $(x + 2)$ is a factor. Divide: $x^3 + 4x^2 + x - 6 = (x+2)(x^2 + 2x - 3) = (x+2)(x+3)(x-1)$. Check: $f(0) = -6$ and $(2)(3)(-1) = -6$ ✓.

Local review guidance: correct theorem applied at the right root; exact quotient and remainder; verification by rebuilding or by f evaluated. This is authored classroom guidance.