Exercise walk-through

4.3 The binomial series for rational n

Pearson Edexcel · International A-Level · Mathematics

The binomial series for any rational n

- Factor: $\sqrt{4+x} = 2\left(1+\frac{x}{4}\right)^{1/2}$, valid $|x| < 4$, with $n = \frac{1}{2}$:
- Check at $x = 1$: series gives $2 + 0.25 - 0.0156 = 2.234$; true $\sqrt{5} = 2.2361\ldots$ ✓ (the third term pulls the two-term value 2.25 down to within 0.002).

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Integration: substitution, parts and partial fractions

- **Substitution** (usually given): rewrite the integral in u, adjust dx via $\frac{du}{dx}$, integrate, substitute back. It is the chain rule reversed.
- **Integration by parts** : $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ — the product rule reversed. Know $\int \ln x\,dx = x\ln x - x + c$ and repeated parts for $\int x^2 e^x dx$-type integrals.

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

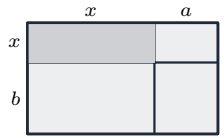
Integration: substitution, parts and partial fractions (2)

- **Partial fractions** turn a **rational** integrand into sums of $\frac{A}{ax+b}$ (integrating to logs) and $\frac{C}{(ax+b)^2}$ (to powers).
- Parts with $u = \mathrm{e}^x$, $\frac{dv}{dx} = \cos x$:
- Parts again on the new integral ($u = \mathrm{e}^x$, $\frac{dv}{dx} = \sin x$):

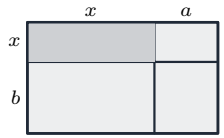
$$= \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x \, dx$$

Exercises 2.1·2.2

2.1 Expand $(1+x)^{\frac{1}{3}}$ to three terms. [3]

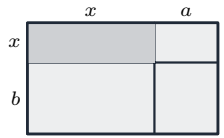


2.2 Expand $(2-x)^{-1}$ to three terms and state the validity. [3]



Exercises 2.1·2.2

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2.1

$$1 + \frac{x}{3} - \frac{x^2}{9} + \cdots$$

Coefficient of x^2: $\frac{1}{3}(-\frac{2}{3}) \cdot \frac{1}{2} = -\frac{1}{9}$.

2.2 Expand $(2-x)^{-1}$ to three terms and state the validity. [3]

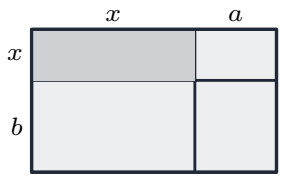


2.2

$$(2-x)^{-1} = \tfrac{1}{2}\left(1-\tfrac{x}{2}\right)^{-1} = \tfrac{1}{2} + \tfrac{x}{4} + \tfrac{x^2}{8}, \qquad |x| < 2.$$

Exercises 2.3·2.4

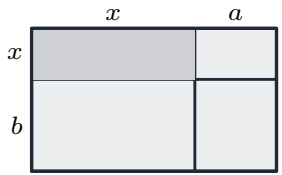
2.3 Expand $\sqrt{9+x}$ to three terms. [3]



2.4 Use your answer to 2.3 to estimate $\sqrt{9.4}$ to 4 decimal places. [2]

Exercises 2.3·2.4

2.3 Expand $\sqrt{9+x}$ to three terms. [3]



2.3

$$\sqrt{9+x} = 3\left(1+\tfrac{x}{9}\right)^{1/2} = 3 + \tfrac{x}{6} - \tfrac{x^2}{216} + \cdots$$

2.4 Use your answer to 2.3 to estimate $\sqrt{9.4}$ to 4 decimal places. [2]

2.4 $x = 0.4$: $3 + 0.0667 - 0.00074 = 3.0659$; true $\sqrt{9.4} = 3.06594$ ✓.

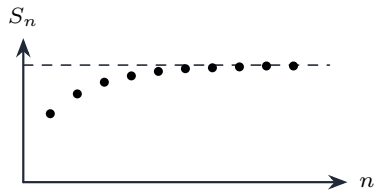
Exercise 3.1

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q8.)*

(a) Find the first three non-zero terms of the binomial series expansion of $\sqrt{4+x}$ in ascending powers of x, giving each coefficient as a simplified fraction.

(b) Write down the first three non-zero terms of the expansion of $\sqrt{4-x}$.

(c) Hence show that $\sqrt{\dfrac{4+x}{4-x}} \approx 1 + \dfrac{x}{4}$, and use it with $x = 1$ to give a rational approximation to one decimal place check of $\sqrt{\frac{5}{3}}$. [7]



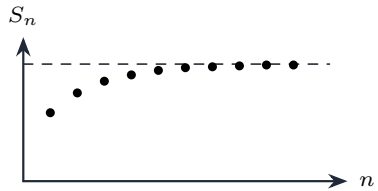
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3.1 (a) From the worked example: $2 + \frac{x}{4} - \frac{x^2}{64}$. (b) Replace x by $-x$: $2 - \frac{x}{4} - \frac{x^2}{64}$. (c) Divide (b)'s expansion into (a)'s:

Full solution on next slide

Full solution 3.1

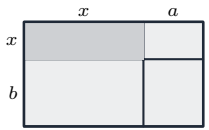
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$$\frac{2 + \frac{x}{4} - \frac{x^2}{64} + \cdots}{2 - \frac{x}{4} - \frac{x^2}{64} + \cdots} = \frac{1 + \frac{x}{8} - \frac{x^2}{128}}{1 - \frac{x}{8} - \frac{x^2}{128}} \approx \left(1 + \frac{x}{8}\right)\left(1 + \frac{x}{8}\right) \approx 1 + \frac{x}{4}$$

(to first order; the x^2 terms cancel to this accuracy: $\frac{x}{8} + \frac{x}{8} = \frac{x}{4}$ and $\frac{x^2}{64} - \frac{x^2}{64} - \frac{x^2}{64} = -\frac{x^2}{64}$, an order-2 remainder). With $x = 1$: $\sqrt{\frac{5}{3}} \approx 1.25$; true value $1.2909\ldots$ – a first-order approximation with visible error, improving with smaller x ✓.

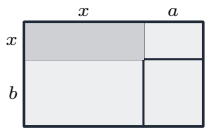
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3.2 Express $\dfrac{7-x}{(2-x)(1+x)}$ in partial fractions and hence expand it in ascending powers of x to three terms, stating the validity. [7]



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3.2 $\dfrac{7-x}{(2-x)(1+x)} = \dfrac{A}{2-x} + \dfrac{B}{1+x}$: $7-x = A(1+x)+B(2-x)$. At $x=2$: $5=3A \Rightarrow A=\frac{5}{3}$. At $x=-1$: $8=3B \Rightarrow B=\frac{8}{3}$.

$$\frac{5}{3}(2-x)^{-1} + \frac{8}{3}(1+x)^{-1} = \frac{5}{6}\left(1-\frac{x}{2}\right)^{-1} + \frac{8}{3}(1+x)^{-1}$$

$$= \frac{5}{6}\left(1+\frac{x}{2}+\frac{x^2}{4}+\cdots\right) + \frac{8}{3}(1-x+x^2-\cdots) = \frac{7}{2} - \frac{9}{4}x + \frac{23}{8}x^2 + \cdots$$

valid for $|x|<1$ (the tighter of $|x|<2$ and $|x|<1$). Check at $x=0$: LHS $=\frac{7}{2}$; series $=\frac{5}{6}+\frac{8}{3}=\frac{7}{2}$ ✓.

Local review guidance: correct n and $\frac{b}{a}$ scaling; simplified fractional coefficients; validity stated and approximations used within it. This is authored classroom guidance.