

Further Pure Mathematics 3

Edexcel IAL FP3 (WFM03/01); the final further pure unit after FP1–FP2 in
Further Mathematics

1. Unit FP3 at a glance

Unit FP3 at a glance

WFM03/01: 1 hour 30 minutes, 75 marks; P1–P4 and FP1 assumed and testable.
Must-know: the P4 dot product and FP1 2×2 matrices, integration by parts and substitution, and quadratic-in- e^x algebra.

2. Hyperbolic functions

Hyperbolic functions

$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\tanh x = \frac{\sinh x}{\cosh x}$. Identities:
 $\cosh^2 x - \sinh^2 x = 1$, $\cosh 2x = 2 \cosh^2 x - 1$. Sum-to-product from the definitions:
 $\cosh 6x + \cosh 4x = 2 \cosh 5x \cosh x$; equations become quadratics in e^x .

3. Inverse hyperbolic functions

Inverse hyperbolics

$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$, $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$, $\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ for $|x| < 1$. Derivation: set $w = \sinh x$, solve $e^{2x} - 2we^x - 1 = 0$, keep the positive root.

4. The ellipse and the hyperbola

Ellipse and hyperbola

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $x = a \cos t$, $y = b \sin t$; $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a \sec t$, $b \tan t$ or $a \cosh t$, $b \sinh t$. Ellipse: $b^2 = a^2(1 - e^2)$; hyperbola: $b^2 = a^2(e^2 - 1)$; foci $(\pm ae, 0)$, directrices $x = \pm \frac{a}{e}$, and $PS = e \cdot PN$.

5. Tangents, normals and loci

Tangents and normals

Tangent condition for $y = mx + c$: $c^2 = a^2m^2 + b^2$ (ellipse) or $a^2m^2 - b^2$ (hyperbola).

Parametrically, differentiate: at $(8 \sec \theta, 7 \tan \theta)$ on $\frac{x^2}{64} - \frac{y^2}{49} = 1$ the tangent is $\frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = 1$, and the QR-diameter circle hits both foci because $qr = -113$ for every θ .

6. Differentiating hyperbolic and inverse functions

Differentiating

$\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$, $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$; inverses give $\frac{1}{\sqrt{x^2+1}}$, $\frac{1}{\sqrt{x^2-1}}$, $\frac{1}{1-x^2}$. For $y = \operatorname{arsinh} x + \operatorname{arsinh} \frac{1}{x}$, $x > 0$: $\frac{dy}{dx} = \frac{x-1}{x\sqrt{1+x^2}}$ — the domain resolves $\sqrt{x^2} = x$.

7. Integrating hyperbolic and inverse functions

Integrating

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c, \quad \int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c, \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c,$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c. \quad \text{Complete the square first: } 4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right).$$

Inverse functions: parts with u the inverse, e.g.

$$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c.$$

8. Reduction formulae, arc length and surfaces

Reduction and surfaces

Parts with the surd as v , then rewrite $\sqrt{f} = \frac{f}{\sqrt{f}}$ to expose I_n : for $I_n = \int_1^6 \frac{x^n}{\sqrt{3x-2}} dx$,
 $(3 + 6n)I_n = 4nI_{n-1} + 8 \cdot 6^n - 2$. Arc length $\int \sqrt{1 + y'^2} dx$; surface $2\pi \int y \sqrt{1 + y'^2} dx$:
 for $y = \cos 2x$ on $[0, \frac{\pi}{4}]$, $2 \sin 2x = \sinh \theta$ gives $S = \frac{\pi}{4}(2\sqrt{5} + \ln(2 + \sqrt{5}))$.

9. Vectors: products, lines and planes

Vectors in 3D

$\mathbf{a} \times \mathbf{b}$ from the **ijk** determinant; $|\mathbf{a} \times \mathbf{b}|$ is the parallelogram area; $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ the volume. Plane $\mathbf{r} \cdot \mathbf{n} = p$ with $\mathbf{n} = \mathbf{b} \times \mathbf{c}$; line $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$; skew distance $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$: for WFM03's planes, $\frac{11}{\sqrt{434}}$.

10. Further matrix algebra

Matrices and eigenvalues

3×3 det by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row

by row: components of $\mathbf{M}\mathbf{v}$ equal λv_j . For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$:

$\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2 . Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T \mathbf{A} \mathbf{P}$ diagonal.

Check yourself

Check yourself

Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x ; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.