Course Companion

Pearson Edexcel · International A-Level · Further Mathematics

Every topic handout in one volume

全部专题讲义合集

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Further Pure Mathematics 1

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Further Pure Mathematics 1: complex numbers, matrices and proof

Further Pure Mathematics 1 (FP1, paper WFM01/01, 1 hour 30 minutes, 75 marks) is compulsory for International AS and A Level Further Mathematics and for International AS and A Level Pure Mathematics. P1–P2 are assumed and may be tested, including sign-change location of roots, rotating shapes about the origin, and dividing a cubic or quartic by a quadratic. This reference covers complex numbers, roots of quadratic equations, numerical methods, the parabola and rectangular hyperbola, matrix algebra, transformations, finite series and proof by induction.

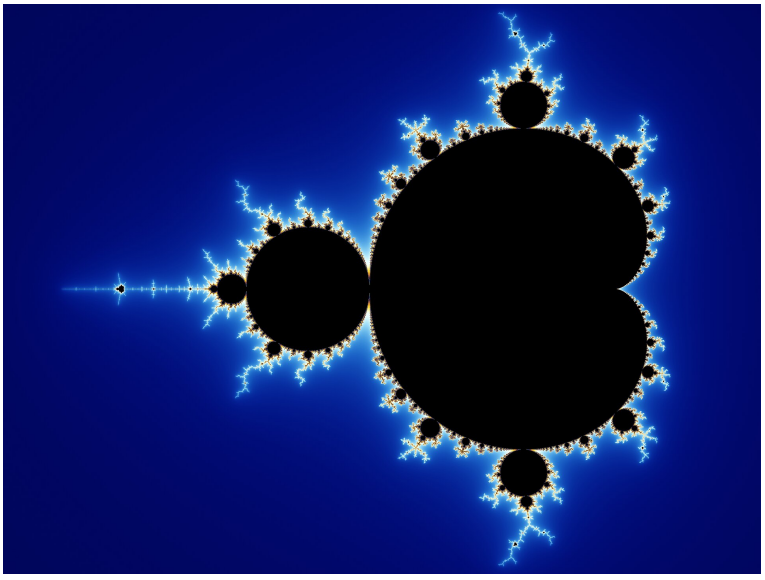
Must-know for FP1 (not in the booklet):

Family	Formulae
quadratic roots	for $ax^2+bx+c=0$ with roots α,β: $\alpha+\beta=-\dfrac{b}{a}, \quad \alpha\beta=\dfrac{c}{a}$
series	$\displaystyle\sum_{r=1}^{n} r=\tfrac{1}{2}n(n+1)$
complex numbers	$z=a+bi$, $\lvert z\rvert=\sqrt{a^2+b^2}$, $\bar{z}=a-bi$, $\lvert z_1z_2\rvert=\lvert z_1\rvert\lvert z_2\rvert$
conics	parabola $y^2=4ax$ (parametric $x=at^2$, $y=2at$); rectangular hyperbola $xy=c^2$ (parametric $x=ct$, $y=\frac{c}{t}$)
matrices	$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-bc, \quad \mathbf{M}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$, $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$

Complex numbers

A **complex number** 复数 is $z=a+bi$ with $i^2=-1$; a is the **real part** 实部, b the **imaginary part** 虚部. The **conjugate** 共轭复数 $\bar{z}=a-bi$ reverses the sign of the imaginary part, and $z\bar{z}=a^2+b^2=|z|^2$ – the trick that makes the denominator real: $\dfrac{z_1}{z_2}=\dfrac{z_1\bar{z}_2}{|z_2|^2}$. Two complex numbers are equal only when both real and imaginary parts match. The polar form $z=r(\cos\theta+i\sin\theta)$ pairs the **modulus** 模 r with the **argument** 辐角 (defined only when $z\neq 0$) θ; multiplying complex numbers multiplies their moduli ($|z_1z_2|=|z_1||z_2|$).

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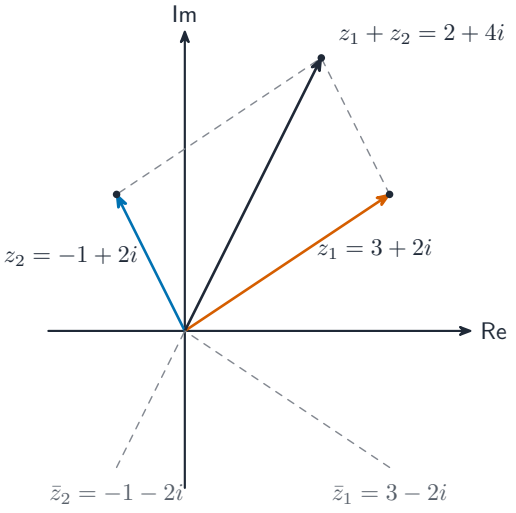


The Mandelbrot set: the complex numbers c for which repeating $z \to z^2 + c$, starting from $z = 0$, stays bounded

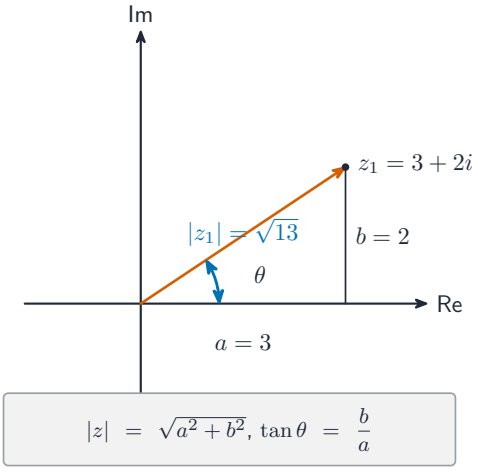
> **Worked example – a real sum pins the imaginary part.** $z_1 = r(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$ with $r > 0$, and z_2 has modulus 5. Since $|z_1 z_2| = |z_1||z_2| = 5r = 15$, we get $r = 3$. So $z_1 = 3(-\frac{\sqrt{3}}{2} - \frac{1}{2}i) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$. If $z_1 + z_2$ is real, its imaginary part is zero, so $z_2 = a + \frac{3}{2}i$; then $a^2 + \frac{9}{4} = 25$ gives $a = \pm\frac{\sqrt{91}}{2} \approx \pm 4.77$. Both $z_2 = \frac{\sqrt{91}}{2} + \frac{3}{2}i$ and its reflection in the imaginary axis work.

On an **Argand diagram** 阿甘特图 (Argand diagram) plot $z = a + bi$ as the point (a, b): addition is vector addition, the conjugate reflects in the real axis, and $|z|$ is the distance from the origin.

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conjugates reflect in the Re axis

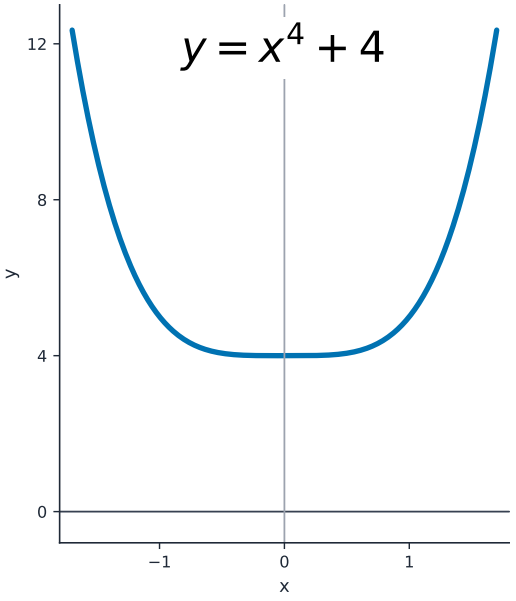


Left: an Argand diagram showing z1 = 3 + 2i, z2 = −1 + 2i, their sum 2 + 4i as a vector parallelogram, and both conjugates dashed below the real axis. Right: the same points with modulus arrows and the argument of z1 marked from the positive real axis.

Complex roots of polynomials

For a quadratic with real coefficients and negative discriminant, complete the square: $x^2 - 6x + 13 = (x - 3)^2 + 4 = 0$ gives $x = 3 \pm 2i$. The roots are a conjugate pair – and this survives degree: **if $f(z_1) = 0$ for real coefficients, then $f(\bar{z}_1) = 0$ too**. So a real cubic has either three real roots or one real root plus one conjugate pair, and $(z - z_1)(z - \bar{z}_1) = z^2 - 2\,\mathrm{Re}(z_1)z + |z_1|^2$ is a real quadratic factor. Divide it out to reach the rest.

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No x-intercepts

Four complex roots

$$1+i, \quad 1-i$$

$$-1+i, \quad -1-i$$

Sum = 0

Product = 4

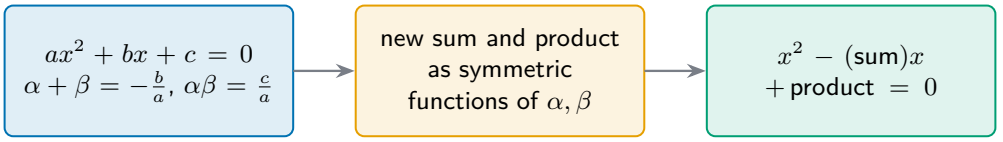
The four roots of $x^4 + 4 = 0$ are $1+i, 1-i, -1+i, -1-i$. Their sum is 0 and their product is 4, although the real graph has no x-intercepts.

> **Worked example – a quartic with a given complex root (WFM01 June 2025 Q7).** $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3+5i$, so $3-5i$ is also a root and
>
> $$z^2 - 6z + 34 \text{ is a factor} \quad (\text{sum } 6,\ \text{product } 9+25=34).$$
>
> Writing $f(z) = (z^2 - 6z + 34)(Pz^2 + bz + c)$ and comparing: $34c = 68$ so $c = 2$; $-6c + 34b = 192$ so $34b = 204$, $b = 6$; then $b - 6P = -36$ gives $P = 7$ and $c - 6b + 34P = Q$ gives $Q = 204$. (Check against $f(3+5i) = 0$: the real equation $644P + 16Q = 7772$ holds, since $644(7) + 16(204) = 4508 + 3264 = 7772$.) The other roots solve $7z^2 + 6z + 2 = 0$:
>
> $$z = \frac{-6 \pm \sqrt{36 - 56}}{14} = \frac{-6 \pm 2i\sqrt{5}}{14} = \frac{-3 \pm i\sqrt{5}}{7}.$$
>
> Without a calculator: only $b^2 - 4ac < 0$ arithmetic and the conjugate fact are used.

Roots of quadratic equations

For $ax^2 + bx + c = 0$ with roots α, β: $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Symmetric expressions in α, β then need no individual roots: $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$, $\alpha^3 + \beta^3 \equiv (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$, and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$. To build a new quadratic, express the new **sum** and **product** in terms of the old ones; the equation is $x^2 - (\text{sum})x + \text{product} = 0$.

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$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

no individual root is ever needed

New roots without solving

Worked example – new roots without solving (WFM01 June 2025 Q6). $f(x) = 3x^2 + kx - 5$ has roots α, β, so $\alpha\beta = -\frac{5}{3}$. If $\alpha + \beta = 9\alpha\beta$, then $-\frac{k}{3} = -15$, so $k = 45$ and $\alpha + \beta = -15$. For roots $\alpha^2 + \beta$ and $\alpha + \beta^2$:

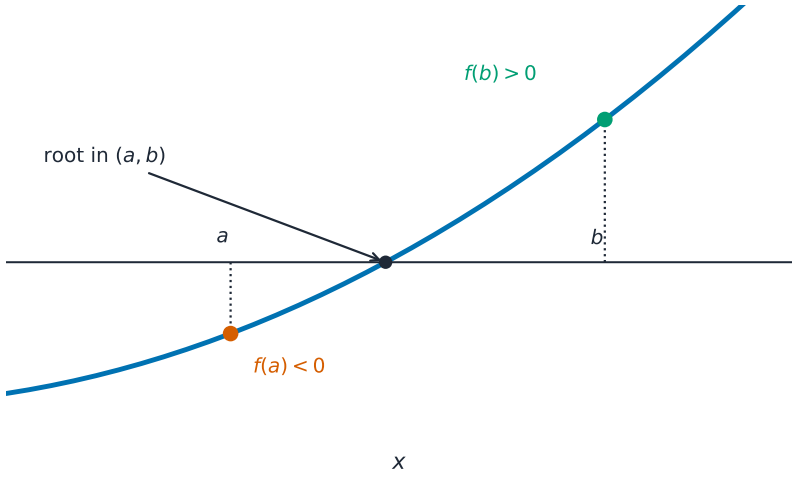
$$\text{sum} = \alpha^2 + \beta^2 + \alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta + (\alpha + \beta) = 225 + \tfrac{10}{3} - 15 = \tfrac{640}{3},$$

$$\text{product} = (\alpha^3 + \beta^3) + \alpha\beta + (\alpha\beta)^2 = -3450 - \tfrac{5}{3} + \tfrac{25}{9} = -\tfrac{31040}{9}.$$

The new equation is $9x^2 - 1920x - 31040 = 0$ (multiply $x^2 - \frac{640}{3}x - \frac{31040}{9} = 0$ by 9).

Numerical solution of equations

Three iterative locators: bisection and interpolation need a continuous function on their bracket; Newton–Raphson also needs a defined, nonzero derivative at each step and can fail to converge. **interval bisection** 区间二分法 halves an interval carrying a sign change; **linear interpolation** 线性插值 chords from $(a, f(a))$ to $(b, f(b))$ and takes the chord's zero $\beta \approx a - \dfrac{(b-a)f(a)}{f(b) - f(a)}$; the **Newton–Raphson method** 牛顿-拉夫逊法 (formula given in the booklet) iterates $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Show every evaluation – "calculator-only" answers score nothing.



if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

For a continuous function on $[a, b]$, opposite endpoint signs trap a root between a and b.

> **Worked example – bisection and interpolation (WFM01 June 2025 Q3).** (i) $f(x) = x^2 + 5 - 8^{5x}$ has a single root α in $[0, 1]$: $f(0) = 4 > 0$, $f(0.5) = 5.25 - 8^{2.5} = -175.77 < 0$, so halve: $f(0.25) = 5.0625 - 8^{1.25} = -8.392 < 0$. The sign change sits in $[0, 0.25]$. (ii) $g(x) = 3^{\sin x} - 3\cos x$ (radians): $g(4) = 2.396 > 0$ and $g(5) = -0.502 < 0$, and g is continuous, so a root β lies in $[4, 5]$. Interpolating once: $\beta \approx 4 + \dfrac{2.396}{2.396 + 0.502} = 4.827$ (4 s.f.).

Newton–Raphson (original example). Solve $x^3 - 2x - 5 = 0$ from $x_0 = 2$: $f(2) = -1$, $f'(x) = 3x^2 - 2$ so $f'(2) = 10$ and $x_1 = 2 - \frac{-1}{10} = 2.1$; then $f(2.1) = 0.061$, $f'(2.1) = 11.23$, $x_2 = 2.0946$. The iterates approach the root, approximately 2.09455 (check: $2.0946^3 - 2(2.0946) - 5 \approx +0.0005$).

Parabola and rectangular hyperbola

The **parabola** 抛物线 $y^2 = 4ax$ has parametric form $x = at^2$, $y = 2at$; the **rectangular hyperbola** 直角双曲线 $xy = c^2$ has $x = ct$, $y = \frac{c}{t}$. Tangents and normals come from differentiating the Cartesian equation (parametric differentiation is not required): $y^2 = 4ax$ gives $\frac{dy}{dx} = \frac{2a}{y} = \frac{1}{t}$ at parameter t, and $xy = c^2$ gives $y = \frac{c^2}{x}$ so $\frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{1}{t^2}$ at $x = ct$.

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A suspension bridge: its main cable hangs close to a parabola, a curve engineers really build

Worked example – normal to a hyperbola (WFM01 June 2025 Q4). For $xy = 81$ at $P(9t, \frac{9}{t})$: $\frac{dy}{dx} = -\frac{81}{x^2} = -\frac{1}{t^2}$, so the normal has gradient t^2:

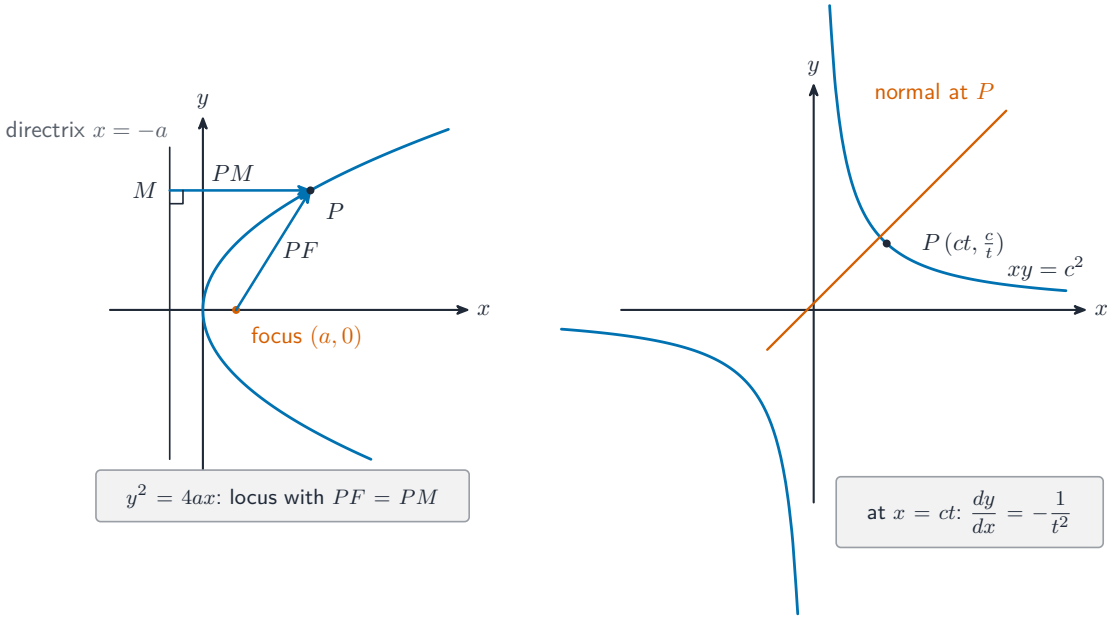
$$y - \tfrac{9}{t} = t^2(x - 9t) \implies ty = t^3x + 9(1 - t^4).$$

At the y-axis, $A = (0, \frac{9(1-t^4)}{t})$. With $t = \frac{1}{3}$: $P = (3, 27)$ and $A = (0, \frac{80}{3})$, so triangle OPA has area $\frac{1}{2} \times 3 \times \frac{80}{3} = 40$.

The parabola is the locus of points equidistant from the **focus** 焦点 $(a, 0)$ and the **directrix** 准线 $x = -a$. Distances, not algebra, unlock the geometry questions.

Worked example – focus-directrix distances (WFM01 June 2025 Q9). P lies on the parabola 5 units from the focus, so P is also 5 units from the directrix ($PF = PM = 5$, with PM perpendicular to the directrix). Points Q, R on the directrix satisfy $PQ = PR = 10$. Each right triangle PQM gives $QM = \sqrt{10^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$, and M lies between Q and R, so $QR = 10\sqrt{3}$.

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Left: the parabola $y^2 = 4ax$ with focus $(a, 0)$, directrix $x = -a$, and a point P whose equal perpendicular distance to focus and directrix ($PF = PM$) is marked with ticks. Right: the rectangular hyperbola $xy = c^2$ with the parametric point $(ct, c/t)$ and the normal line at that point.

Matrix algebra

Matrices add entry-wise (same shape), scale entry-wise, and multiply row-by-column ($(\mathbf{AB})_{ik} = \sum_j A_{ij}B_{jk}$; $\mathbf{AB} \neq \mathbf{BA}$ in general). For 2×2 matrices $\det \mathbf{M} = ad - bc$: zero means **singular** 奇异 (no inverse); otherwise $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$. Inverses reverse order: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ – undo the last transformation first.

Worked example – an inverse that must stay consistent (WFM01 June 2025 Q1). $\mathbf{M} = \begin{pmatrix} 1 & a \\ 3 & -5 \end{pmatrix}$ with $a \neq -\frac{5}{3}$:

$$\mathbf{M}^{-1} = \frac{1}{-(5+3a)}\begin{pmatrix} -5 & -a \\ -3 & 1 \end{pmatrix}.$$

Given $\mathbf{M}^{-1} = 2\mathbf{M} + 8\mathbf{I} = \begin{pmatrix} 10 & 2a \\ 6 & -2 \end{pmatrix}$, compare entries: $\dfrac{-3}{-(5+3a)} = 6$ gives $5 + 3a = \frac{1}{2}$, and every other entry confirms it, so $a = -\frac{3}{2}$.

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Transformations using matrices

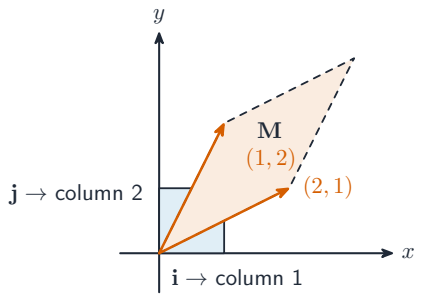
A 2×2 matrix $\mathbf{M}$ transforms column vectors by $\mathbf{y}=\mathbf{Mx}$; its **columns are the images of i and j** – read any transformation straight off the matrix. The standard results:

Transformation	Matrix
reflection in the x-axis	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
reflection in the y-axis	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$
reflection in $y=x$ / $y=-x$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ / $\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$
rotation, angle θ anticlockwise about O	$\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$
stretch, factor k parallel to the x-axis	$\begin{pmatrix}k&0\\0&1\end{pmatrix}$ (swap for the y-axis)
enlargement, scale factor k about O	$\begin{pmatrix}k&0\\0&k\end{pmatrix}$

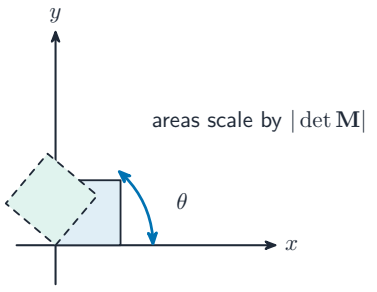
For combinations, **AB** means "apply **B** first, then **A**". The inverse transformation undoes it, and $|\det\mathbf{M}|$ is the **area scale factor** 面积比例因子: areas multiply by the determinant (negative determinants add a reflection).

> **Worked example – describing and combining (WFM01 June 2025 Q2).** $\mathbf{P}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ maps $\mathbf{i}\to(0,-1)^T$ and $\mathbf{j}\to(1,0)^T$: a quarter-turn clockwise (270° anticlockwise) about O. $\mathbf{Q}=5\mathbf{I}$ is an enlargement, scale factor 5, centre O. Reflection in $y=-x$ is $\mathbf{R}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$, so $A=\mathbf{R}(4,3)^T=(-3,-4)$. If $\mathbf{Q}$ follows $\mathbf{P}$ and maps B to A, then $B=\mathbf{P}^{-1}\mathbf{Q}^{-1}A=\mathbf{P}^{-1}\cdot\frac{1}{5}(-3,-4)^T=\frac{1}{5}(4,-3)^T=(0.8,-0.6)$ – $\mathbf{P}$'s own inverse is its transpose, so this is arithmetic only.

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$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: columns are the images of $\mathbf{i}, \mathbf{j}$



rotation by θ: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Left: the unit square with basis vectors i and j, and its image under the matrix with columns equal to the images of i and j. Right: the same transformation shown as a rotation and stretch combination, with the determinant marked as the area scale factor between the two parallelograms.

Series

With $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (and $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$ from the booklet), any polynomial summand splits into standard pieces – including sums to an even limit $2n$ (substitute $k = 2n$ into the standard formulae) and shifted sums like $\sum(r+1)^2$.

> **Worked example – a sum to 2n (WFM01 June 2025 Q8a).** Show $\sum_{r=1}^{2n}(2r^2 - 1) = \frac{4}{3}n(n+1)(an+b)$:

$$2\sum_{1}^{2n} r^2 - 2n = 2 \cdot \frac{(2n)(2n+1)(4n+1)}{6} - 2n = \frac{2n(2n+1)(4n+1) - 6n}{3} = \frac{2n(8n^2+6n-2)}{3} = \frac{4n(4n^2+3n-1)}{3},$$

and $4n^2 + 3n - 1 = (n+1)(4n-1)$, so the sum is $\frac{4}{3}n(n+1)(4n-1)$: $a = 4$, $b = -1$. (Check $n = 1$: $1 + 7 = 8$ and $\frac{4}{3} \cdot 1 \cdot 2 \cdot 3 = 8$.)

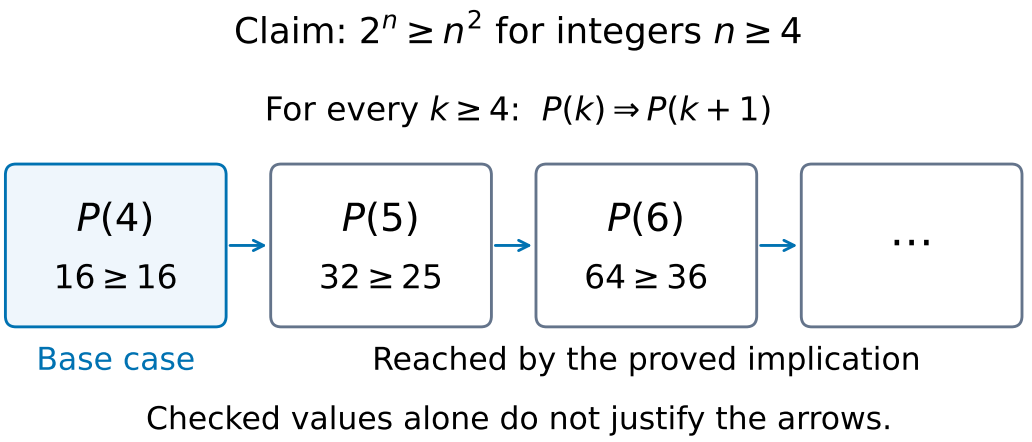
Proof by mathematical induction

Induction 数学归纳法 proves a statement $P(n)$ for all positive integers in three moves: verify $P(1)$; assume $P(k)$; deduce $P(k+1)$ from it – then close with "true for $n = 1$, and truth for $n = k$ implies truth for $n = k + 1$, so by induction true for all positive integers n". The closing sentence is part of the proof. Four assessed families: summation formulae, divisibility, sequences given by a recurrence, and matrix powers.

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Falling dominoes: prove the first case, prove each case knocks over the next, and every case falls



Worked example – a summation (WFM01 June 2025 Q8b). Prove $\sum_{r=1}^{n} r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$. For $n=1$: LHS $= 1\cdot 1^2 = 1$ and RHS $= \frac{1}{4}\cdot 1\cdot 2\cdot 2 = 1$ – true. Assume true for $n=k$; the $(k+1)$th term is $(k+1)(3k+1)^2$, so

$$\sum_{r=1}^{k+1} r(3r-2)^2 = \tfrac{1}{4}k^2(k+1)(9k-7) + (k+1)(3k+1)^2 = \tfrac{1}{4}(k+1)\left[k^2(9k-7) + 4(3k+1)^2\right].$$

Expand the bracket: $9k^3 - 7k^2 + 36k^2 + 24k + 4 = 9k^3 + 29k^2 + 24k + 4$. The target for $n = k+1$ is $\frac{1}{4}(k+1)^2(k+2)(9k+2)$, and its bracket $(k+1)(k+2)(9k+2)$ expands as $(k^2+3k+2)(9k+2) = 9k^3 + 29k^2 + 24k + 4$ – identical. So the statement holds for $n = k+1$, and by induction for all positive integers n.

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Divisibility (original example). Prove $n^3 + 2n$ is divisible by 3. For $n = 1$: 3. Assume $k^3+2k = 3m$; then $(k+1)^3+2(k+1) = (k^3+2k)+3k^2+3k+3 = 3(m+k^2+k+1)$, still a multiple of 3. By induction, true for all n.

Matrix powers (original example). Prove $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$. True for $n = 1$. If true for k, then $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & k+1\\ 0 & 1\end{pmatrix}$, completing the induction.

Checklist before you call this unit done

- Work with $a + bi$ and $r(\cos\theta + i\sin\theta)$: modulus, argument, conjugate; quotients made real.
- Use conjugate pairs to factor cubics and quartics with real coefficients; find all roots without a calculator.
- Convert symmetric expressions via $\alpha + \beta$ and $\alpha\beta$; form new quadratics for transformed roots.
- Locate roots by bisection and linear interpolation with every evaluation shown; iterate Newton–Raphson from a given start.
- Use $y^2 = 4ax$ and $xy = c^2$ parametrics; tangents/normals via Cartesian differentiation; focus-directrix distances.
- Invert 2×2 matrices (and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$); solve matrix equations entry by entry.
- Identify reflections, rotations, stretches, enlargements from matrices; combine in the right order; use $|\det|$ as the area scale factor.
- Sum series to n and $2n$ with the standard formulae; prove series, divisibility, recurrence and matrix results by induction with a closing statement.

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Further Pure Mathematics 2

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Further Pure Mathematics 2: inequalities, further complex numbers and differential equations

Further Pure Mathematics 2 (FP2, paper WFM02/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) builds directly on P1–P4 and FP1 and counts one sixth of an International Advanced Level. Every unit is equally weighted, so FP2 material can also surface inside Further Mathematics. P1–P4 assumed content stays live: partial fractions, parametric and implicit differentiation, integration by parts and substitution, and the whole FP1 complex-number and proof toolkit. This reference covers modulus inequalities, the method of differences, Euler's relation and de Moivre's theorem, loci and z→w transformations, first and second order differential equations, Maclaurin and Taylor series, and polar coordinates.

Must-know for FP2 (assumed, not re-taught):

Family	Formulae
modulus	$\lvert x\rvert = x$ for $x \geq 0$ and $-x$ for $x < 0$; $\lvert z\rvert = \sqrt{a^2+b^2}$, $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
sums	$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum r^2$, $\sum r^3$ (booklet)
algebra	partial fractions, e.g. $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$ (P4)
calculus	product, quotient and chain rules; integration by substitution, parts, partial fractions (P4)
differential equations	separation of variables; exponential growth/decay $y = Ae^{kt}$ (P4)

Inequalities involving the modulus sign

Two safe moves and one banned move. Safe: split into cases from the definition of the modulus ($\lvert f\rvert = f$ where $f \geq 0$, $-f$ where $f < 0$), and square both sides of $\lvert A\rvert > \lvert B\rvert$ to get $A^2 > B^2$. Banned: multiplying or dividing by an expression whose sign you have not fixed – case-split on that sign instead, and intersect each case with its own domain.

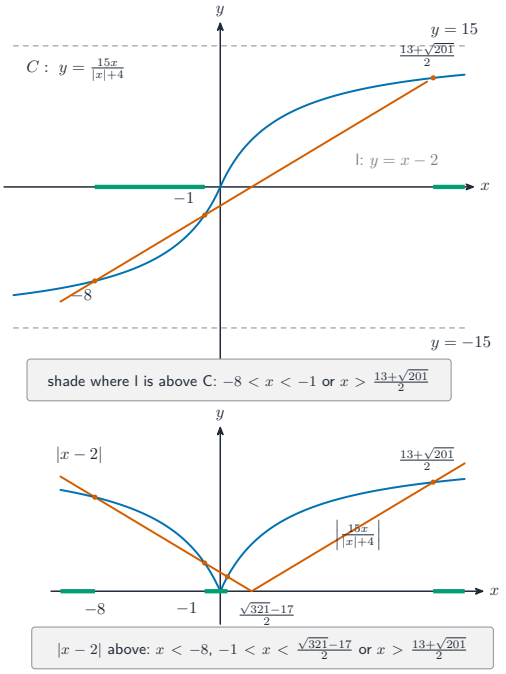
> **Worked example – one branch dies (specimen style).** Solve $\lvert x^2-1\rvert > 2(x+1)$. Where $x^2 \geq 1$: $x^2-2x-3 > 0$, so $(x-3)(x+1) > 0$, giving $x > 3$ or $x < -1$; keeping only $x \leq -1$ or $x \geq 1$ leaves $x < -1$ or $x > 3$. Where $x^2 < 1$: $1 - x^2 > 2x + 2$ becomes $(x+1)^2 < 0$ – impossible. The solution is $x < -1$ or $x > 3$. A dead branch is a result, not a mistake: report it.

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Worked example – modulus on both sides (WFM02 June 2025 Q8). The curve C has equation $y = \dfrac{15x}{|x|+4}$ and the line l has equation $y = x - 2$.

(a) Solve $x - 2 > \dfrac{15x}{|x|+4}$. The denominator is always positive, so multiply straight through. For $x \geq 0$: $(x-2)(x+4) > 15x$ gives $x^2 - 13x - 8 > 0$, so $x > \frac{13+\sqrt{201}}{2}$ (the other root is negative). For $x < 0$: $(x-2)(4-x) > 15x$ gives $-x^2 - 9x - 8 > 0$, i.e. $(x+1)(x+8) < 0$, so $-8 < x < -1$. Together: $-8 < x < -1$ or $x > \frac{13+\sqrt{201}}{2}$.

(b) Solve $|x-2| > \left|\dfrac{15x}{|x|+4}\right|$. Both sides are non-negative, so square and multiply by $(|x|+4)^2$: the inequality becomes $PQ > 0$, with $P = (x-2)(|x|+4) - 15x$ (the expression from part (a)) and $Q = (x-2)(|x|+4) + 15x$. So P and Q must have the **same sign**. It is not enough to have "$P > 0$ or $Q < 0$". For $x \geq 0$: $P = x^2 - 13x - 8$ and $Q = x^2 + 17x - 8$. Both are positive for $x > \frac{13+\sqrt{201}}{2}$, and both are negative for $0 \leq x < \frac{\sqrt{321}-17}{2}$. For $x < 0$: $Q = -x^2 + 21x - 8$ is negative for every $x < 0$, so we need $P < 0$, which is $(x+1)(x+8) > 0$: $x < -8$ or $-1 < x < 0$. Together: $x < -8$ or $-1 < x < \frac{\sqrt{321}-17}{2}$ or $x > \frac{13+\sqrt{201}}{2}$. (Check $x = -5$: $|x-2| = 7$ but $\left|\frac{-75}{9}\right| = 8.3$, so -5 is correctly excluded.) The "hence" is real: part (a)'s factor P does half the work.

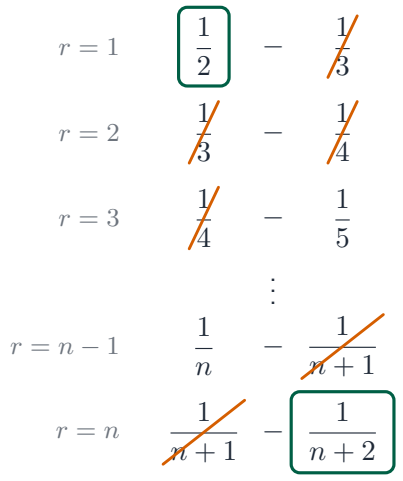


Top: the curve $y = 15x/(|x| + 4)$ with its horizontal asymptotes at ± 15, the line $y = x - 2$, and the intervals where the line is above the curve shaded. Below: both sides' moduli – the reflected curve $|15x/(|x| + 4)|$ and the V of $|x - 2|$, which cross at $x = -8$, -1, $(\sqrt{321} - 17)/2$ and $(13 + \sqrt{201})/2$, with the three solution intervals shaded.

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The method of differences

The **method of differences** 差分法 sums a series by splitting the summand (usually via **partial fractions** 部分分式) so that each term cancels part of a later one – the middle collapses and only the first and last fragments survive. Write out $r = 1$, $r = 2$, $r = n - 1$, $r = n$ explicitly and watch what cancels; that display earns the marks.



each struck $-$ term cancels the $+$ term on the next line; only the boxed first and last survive:

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$$

The method of differences: write out the first and last rows and watch the middle cancel

Worked example – telescoping to a closed form (WFM02 June 2025 Q5). Show $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$. Split:

$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}.$$

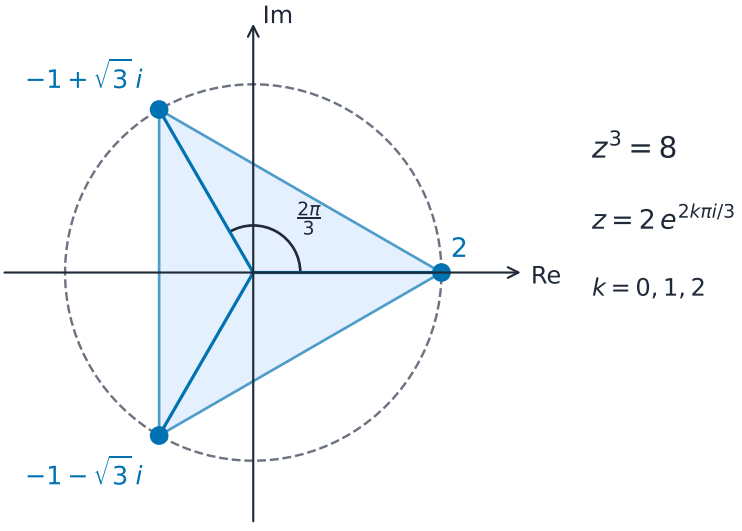
The sum is $\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$: everything from $\frac{1}{3}$ to $\frac{1}{n+1}$ cancels, leaving $\frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$. (Check $n = 1$: both sides $\frac{1}{6}$.) This is also how inequality follow-ups are set: $\dfrac{n}{2(n+2)} > 0.49$ needs $n > 98$, so the smallest is $n = 99$ – the algebra, not a table of values.

Euler's relation and de Moivre's theorem

Euler's relation 欧拉关系式 $e^{i\theta} = \cos\theta + i\sin\theta$ turns the polar form into an exponential: $z = re^{i\theta}$, and $z^n = r^n e^{in\theta}$ becomes pure arithmetic. Read off the companions $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$. **De Moivre's theorem** 棣莫弗定理 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ follows, and is provable for every positive integer n by induction: true for $n = 1$; if true for k, multiplying $(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

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and expanding with the addition formulae gives $\cos(k+1)\theta + i\sin(k+1)\theta$. For negative n, take reciprocals and note $\cos(-n\theta) + i\sin(-n\theta)$ has modulus 1 and the right argument.



The nth roots are equally spaced round a circle

Worked example – a multiple-angle identity (WFM02 June 2025 Q9). Expand the imaginary part of $(\cos\theta + i\sin\theta)^5$ by the binomial theorem:

$$\text{Im} = 5c^4 s - 10c^2 s^3 + s^5 \quad (c = \cos\theta,\ s = \sin\theta).$$

De Moivre says this equals $\sin 5\theta$. Substituting $c^2 = 1 - s^2$: $5s(1-s^2)^2 - 10s^3(1-s^2) + s^5 = 16s^5 - 20s^3 + 5s$, so

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Setting $5\theta = \pi$ gives $16s^4 - 20s^2 + 5 = 0$, so $s^2 = \frac{5\pm\sqrt{5}}{8}$ and $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$ – take the smaller root because $\sin\frac{\pi}{5} < \sin\frac{3\pi}{5}$, and indeed $\sin\frac{3\pi}{5}$ is the larger root.

Worked example – fifth roots (WFM02 June 2025 Q1). Solve $z^5 = -2\sqrt{2} - 2\sqrt{6}i$. Modulus $\sqrt{8+24} = \sqrt{32} = 4\sqrt{2}$; the argument is $-\frac{2\pi}{3}$ (third quadrant, $\tan = \sqrt{3}$). So

$$z = (4\sqrt{2})^{1/5}\, e^{i(-2\pi/15 + 2\pi k/5)} = \sqrt{2}\, e^{i\left(-\frac{2\pi}{15} + \frac{2\pi k}{5}\right)}, \quad k = 0, 1, 2, 3, 4,$$

five roots spaced $\frac{2\pi}{5}$ apart on the circle $|z| = \sqrt{2}$. Use the nth-root pattern $\sqrt[n]{r}\, e^{i(\theta+2\pi k)/n}$ – never decimal approximations.

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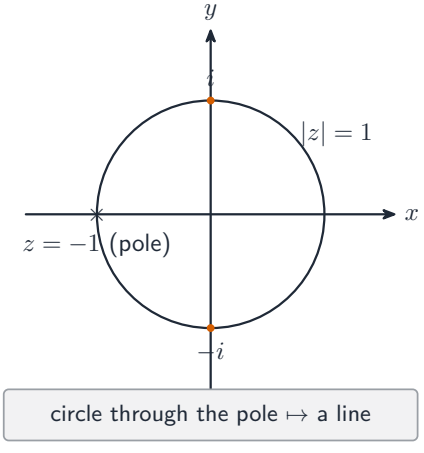
Loci, regions and z→w transformations

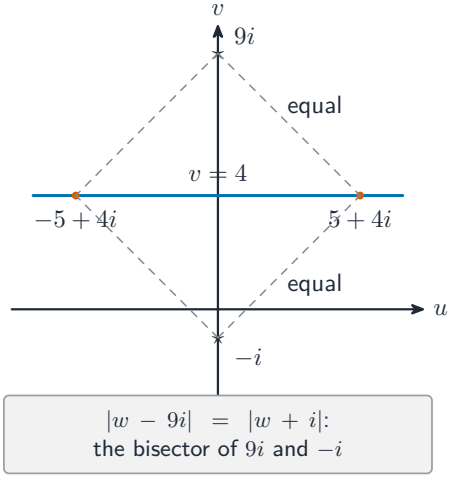
Locus 轨迹 toolkit in the Argand diagram: $|z-a|=b$ is a circle (centre a, radius b); $|z-a|=|z-b|$ is the **perpendicular bisector** 垂直平分线 of a and b; $|z-a|=k|z-b|$ with $k\neq 1$ is a circle (square both sides and collect terms); $\arg(z-a)=\beta$ is a half-line from a. Regions replace $=$ by $<$ or $>$: test one convenient point (the origin rarely lies on the locus). For **Möbius transformations** 莫比乌斯变换 $w=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$, first choose the useful algebraic form rather than relying only on substitution point-by-point: make z the subject first, then take moduli (or arguments) of both sides. When $c\neq 0$, a circle through the pole $z=-\frac{d}{c}$ maps to a straight line; a circle not through the pole maps to a circle.

> **Worked example – image of a circle (WFM02 June 2025 Q3).** $w=\dfrac{9i-iz}{1+z}$ $(z\neq -1)$ maps P, where $|z|=1$, to a line l. Making z the subject: $z(w+i)=9i-w$, so $z=\dfrac{9i-w}{w+i}$. Then $|z|=1$ forces

$$|9i-w|=|w+i| \implies u^2+(v-9)^2=u^2+(v+1)^2 \implies 81-18v=2v+1,$$

so $20v=80$ and l is $v=4$ – the perpendicular bisector of $9i$ and $-i$, exactly as the equal-moduli reading predicts. Spot-check with the endpoints of the vertical diameter: $z=i\mapsto 5+4i$ and $z=-i\mapsto -5+4i$; both land on $v=4$.





Left: the z-plane with the unit circle $|z|=1$ and the pole $z=-1$ marked. Right: the w-plane with the image line $v=4$ and the images of $z=i$ and $z=-i$ lying on it.

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First order differential equations

Separable equations are assumed. The new standard form is $\dfrac{dy}{dx} + Py = Q$ with P, Q functions of x: multiply by the **integrating factor** 积分因子 $e^{\int P\,dx}$, chosen so the left side becomes $\dfrac{d}{dx}\left(ye^{\int P\,dx}\right)$, then integrate once. For a **given substitution** (say $z = f(y)$), compute $\dfrac{dz}{dx}$ by the chain rule, rewrite the equation in z and x, solve the (now linear or separable) equation, and substitute back – the question names the target form, so end in it.



A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Worked example – integrating factor warm-up. $\dfrac{dy}{dx} + 2y = e^{-x}$: $e^{\int 2\,dx} = e^{2x}$, so $\dfrac{d}{dx}(ye^{2x}) = e^{x}$ and $ye^{2x} = e^{x} + C$, i.e. $y = e^{-x} + Ce^{-2x}$.

Worked example – a substitution then an integrating factor (WFM02 June 2025 Q6). Equation (I) is $2\dfrac{dy}{dx} + y\cot x + y^{3}\tan x\sec x = 0$ on $0 < x < \frac{\pi}{2}$.

(a) With $z = \frac{1}{y^{2}}$, $\dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$, so $2\dfrac{dy}{dx} = -y^{3}\dfrac{dz}{dx}$. Substituting and dividing by y^{3}: $-\dfrac{dz}{dx} + y^{-2}\cot x + \tan x\sec x = 0$, i.e.

$$\frac{dz}{dx} - z\cot x = \tan x\sec x \quad \text{(II)}.$$

(b) The integrating factor is $e^{-\int \cot x\,dx} = \csc x$, so $\dfrac{d}{dx}(z\csc x) = \csc x\tan x\sec x =$

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$\frac{1}{\cos^2 x} = \sec^2 x$. Hence $z \csc x = \tan x + C$, giving $z = \sin x \tan x + C \sin x$ and

$$y^2 = \frac{1}{\sin x \tan x + C \sin x}.$$

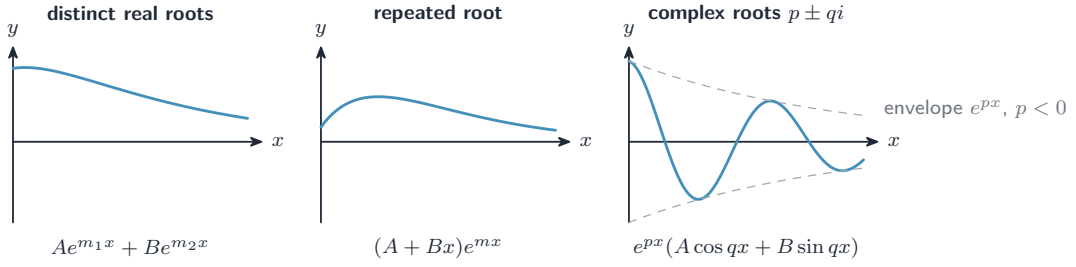
(c) At $x = \frac{\pi}{6}$: $y^2 = \frac{4\sqrt{3}}{3}$ so $z = \frac{\sqrt{3}}{4} = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} + \frac{C}{2}$, forcing $C = \frac{1}{2\sqrt{3}}$. At $x = \frac{\pi}{3}$: $z = \frac{\sqrt{3}}{2} \cdot \sqrt{3} + \frac{1}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$, so $y^2 = \frac{4}{7}$ and $y = \pm\frac{2}{\sqrt{7}}$. Both signs are answers – y^2, not y, was determined.

Second order differential equations

For $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, the general solution is CF + PI: the **complementary function** 余函数 solves the homogeneous equation via the **auxiliary equation** 辅助方程 $am^2 + bm + c = 0$; the **particular integral** 特积分 is any single solution of the full equation.



A pendulum: small swings obey a second-order differential equation, simple harmonic motion



The roots of the auxiliary equation set the shape of the complementary function; here every root has a negative real part.

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Auxiliary equation	Complementary function
distinct real roots $m_1 \neq m_2$	$y = Ae^{m_1 x} + Be^{m_2 x}$
repeated root m	$y = (A + Bx)e^{mx}$
complex roots $p \pm qi$	$y = e^{px}(A\cos qx + B\sin qx)$

Right-hand side $f(x)$	Trial PI
ke^{px}	λe^{px}; if p is a root, use $\lambda x e^{px}$ (again if repeated)
$a + bx(+cx^2)$	$\lambda + \mu x(+\nu x^2)$ – every degree present
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

Given initial or boundary conditions, find the constants last, from the full general solution.

> **Worked example – resonance, then conditions (WFM02 June 2025 Q4).** Given that $y = \lambda x e^{3x}$ is a particular integral of $4\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 3y = 78e^{3x}$:

(a) $y' = \lambda e^{3x}(1 + 3x)$ and $y'' = \lambda e^{3x}(6 + 9x)$, so the left side is $\lambda e^{3x}\big(24 + 36x - 11 - 33x - 3x\big) = 13\lambda e^{3x}$. Matching $78e^{3x}$ gives $\lambda = 6$. (The plain trial λe^{3x} would give $0 \cdot e^{3x}$ – hence the extra x.)

(b) Auxiliary equation $4m^2 - 11m - 3 = (4m + 1)(m - 3) = 0$, so $m = -\frac{1}{4}$ or 3: $y = Ae^{-x/4} + Be^{3x} + 6xe^{3x}$.

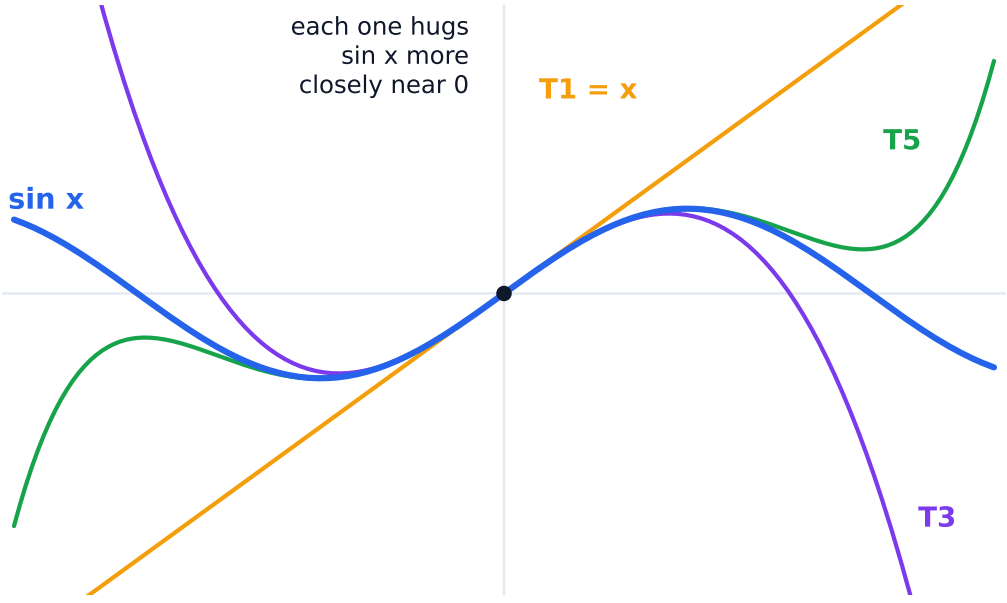
(c) With $y = \frac{9}{2}$ and $y' = 0$ at $x = 0$: $A + B = \frac{9}{2}$ and $-\frac{A}{4} + 3B + 6 = 0$. Solving, $A = 6$, $B = -\frac{3}{2}$, so

$$y = 6e^{-x/4} - \tfrac{3}{2}e^{3x} + 6xe^{3x}.$$

Maclaurin and Taylor series

The **Maclaurin series** 麦克劳林级数 of f is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ – differentiate as many times as the question asks (third and higher derivatives are examinable in their own right). The **Taylor series** 泰勒级数 about $x = a$ replaces x by the offset: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$. For a differential equation, differentiate the equation itself to climb the derivative ladder: $y(a)$ and $y'(a)$ feed the equation for $y''(a)$, then the derivative of the equation for $y'''(a)$, and so on – never re-derive, differentiate.

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The curves compare $\sin x$ *with* $T_1 = x$, $T_3 = x - x^3/6$ *and* $T_5 = x - x^3/6 + x^5/120$.

Worked example – a series solution about a non-zero centre (WFM02 June 2025 Q2). Given $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = e^{2x/9}$ with $y = 2$ and $y' = -1$ at $x = 3$:

at $x = 3$: $y'' = 4y' + 5y + e^{2/3} = -4 + 10 + e^{2/3} = 6 + e^{2/3}$, so the quadratic term is $\frac{1}{2}(6 + e^{2/3})(x-3)^2 = (3 + \frac{1}{2}e^{2/3})(x-3)^2$. Differentiating the equation, $y''' = 4y'' + 5y' + \frac{2}{9}e^{2x/9}$, so at $x = 3$: $y''' = 4(6 + e^{2/3}) - 5 + \frac{2}{9}e^{2/3} = 19 + \frac{38}{9}e^{2/3}$, and the cubic term is $\frac{1}{6}y'''(x-3)^3$. The series is

$$y = 2 - (x-3) + \left(3 + \tfrac{1}{2}e^{2/3}\right)(x-3)^2 + \left(\tfrac{19}{6} + \tfrac{19}{27}e^{2/3}\right)(x-3)^3.$$

Polar coordinates and areas

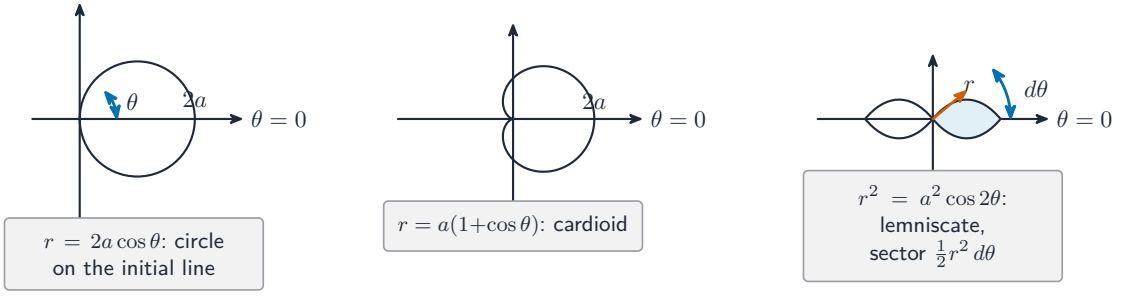
In **polar coordinates** 极坐标 a point is (r, θ) with $r \geq 0$ measured from the **initial line** 初始线 (the positive x-axis, θ anticlockwise). Convert with $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$: for example $r = 4\cos\theta$ gives $r^2 = 4r\cos\theta$, i.e. $x^2 + y^2 = 4x$ – the circle $(x-2)^2 + y^2 = 4$. Know the sketch staples: $r = 2a\cos\theta$ (circle on the initial line through the pole), $r = k\theta$ (spiral), $r = a(1 + \cos\theta)$ (**cardioid** 心形线), $r = a\cos 2\theta$ (four petals), $r^2 = a^2\cos 2\theta$ (**lemniscate** 双纽线, existing only where $\cos 2\theta \geq 0$). The area swept from $\theta = \alpha$ to β is $\frac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$; tangents parallel to the initial line occur where $\dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta = 0$.

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A spiral staircase: spirals are described most simply in polar coordinates

Worked example — exact areas. The whole **lemniscate** $r^2 = 8\cos 2\theta$: by symmetry the right loop is $\frac{1}{2}\int_{-\pi/4}^{\pi/4} 8\cos 2\theta\, d\theta = 4\left[\frac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = 4$, so both loops enclose total area 8. The **cardioid** $r = 2(1+\cos\theta)$: $\frac{1}{2}\int_{-\pi}^{\pi} 4(1+\cos\theta)^2 d\theta = 2\int_{-\pi}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta = 2(2\pi+0+\pi) = 6\pi$, since $\int_{-\pi}^{\pi}\cos^2\theta\, d\theta = \pi$.



Three polar staples drawn on polar grids: the circle $r = 2a\cos\theta$ through the pole, the cardioid $r = a(1+\cos\theta)$, and the lemniscate $r^2 = a^2\cos 2\theta$ with one loop shaded and the sector area element $\frac{1}{2}r^2\,d\theta$ marked.

Checklist before you call this unit done

- Solve modulus inequalities by case split and by squaring; handle $|x|$ inside denominators; report dead branches; reuse an earlier part when "hence" appears.
- Sum finite series by partial fractions and cancellation, writing the $r = 1, 2, n-1, n$ lines; close the follow-up inequality by algebra.

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- Use $e^{i\theta}$, the exponential forms of cos and sin, and de Moivre (proved for integer n) to expand multiple angles and powers, and to find all nth roots in exact polar form.
- Sketch circles, bisectors, Apollonius circles and argument half-lines; test regions; make z the subject of a Möbius transformation and equate moduli or arguments.
- Solve $\frac{dy}{dx} + Py = Q$ with an integrating factor; execute given substitutions and answer in the form demanded, minding $\pm$ from y^2.
- Solve second order equations via auxiliary equation and trial PI (including the multiply-by-x resonance), then fit initial conditions.
- Build Maclaurin and Taylor series, including series solutions of differential equations by differentiating the equation.
- Sketch polar staples, convert to and from Cartesian, and evaluate exact areas with $\frac{1}{2}\int r^2\,d\theta$.

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Further Pure Mathematics 3

Pearson Edexcel · International A-Level · Further Mathematics

Further Pure Mathematics 3: hyperbolic functions, conics, vectors and matrices

Further Pure Mathematics 3 (FP3, paper WFM03/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the last pure unit: P1–P4 and FP1 are assumed and may be tested. It adds the hyperbolic functions and their inverses, the ellipse and hyperbola with their focus-directrix properties, the remaining differentiation and integration techniques (reduction formulae, arc length, surfaces of revolution), vector products with lines and planes in three dimensions, and matrix algebra up to eigenvalues and diagonalisation. This reference covers all of it, with the June 2025 paper as the worked spine.

Must-know for FP3 (assumed, not re-taught):

Family	Formulae
calculus	product, quotient and chain rules; integration by parts and by substitution (P4, FP2)
trigonometry	$\sec^2\theta \equiv 1+\tan^2\theta$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, double angles
vectors (P4)	$\mathbf{a}\cdot\mathbf{b} = a_1b_1+a_2b_2+a_3b_3$, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$
matrices (FP1)	2×2 inverse, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, transformation matrices
exponentials	$e^{\ln x} = x$, quadratic-in-e^x substitutions, ln laws

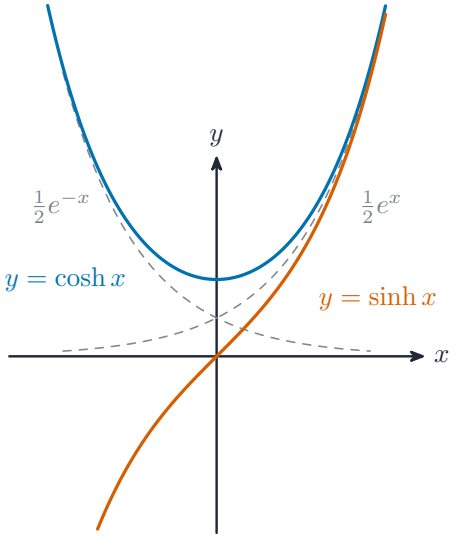
Hyperbolic functions

The **hyperbolic functions** 双曲函数 are built from exponentials: $\cosh x = \frac{1}{2}(e^x+e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, with $\tanh x = \dfrac{\sinh x}{\cosh x}$. The graph of cosh is the **catenary** 悬链线, even with minimum $(0,1)$; sinh is odd through the origin; tanh is odd and trapped between the **asymptote** 渐近线 lines $y=\pm 1$. The two identities to derive on demand are $\cosh^2 x - \sinh^2 x \equiv 1$ and $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$; the sum-to-product identities mirror the trigonometric ones.

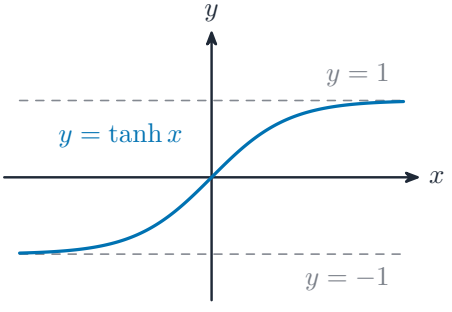
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A chain hanging between two posts sags into a catenary, the curve $y = a\cosh(x/a)$



$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x}),\quad \sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$



$$\tanh x = \frac{\sinh x}{\cosh x},\ \text{bounded by } \pm 1$$

Left: $y = \cosh x$ built from the dashed half-exponentials, with $y = \sinh x$. Right: $y = \tanh x$ between its asymptotes $y = \pm 1$.

Worked example – an identity from the definitions (WFM03 June 2025 Q1). Show that $\cosh 6x + \cosh 4x = 2\cosh 5x \cosh x$. From the left:

$$\tfrac{1}{2}(e^{6x} + e^{-6x}) + \tfrac{1}{2}(e^{4x} + e^{-4x}) = \tfrac{1}{2}(e^{5x} + e^{-5x})(e^{x} + e^{-x}) = 2\cosh 5x \cosh x,$$

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pairing $e^{6x} = e^{5x}e^{x}$ with $e^{-6x} = e^{-5x}e^{-x}$ and likewise for the $4x$ terms – every step is bookkeeping with products $e^{\pm x}e^{\pm 5x}$, which is why the marks demand the exponential forms before any cancellation. Then solve $\cosh 6x + \cosh 4x = 8\cosh x$: the identity collapses it to $\cosh x(2\cosh 5x - 8) = 0$, and cosh is never zero, so $\cosh 5x = 4$, giving $5x = \pm\operatorname{arcosh} 4 = \pm\ln(4+\sqrt{15})$ and $x = \pm\frac{1}{5}\ln(4+\sqrt{15})$.

> **Worked example – a cosh–sinh equation (original).** Solve $2\cosh x + \sinh x = 2$: the exponential definitions give $2\cdot\frac{1}{2}(e^{x}+e^{-x}) + \frac{1}{2}(e^{x}-e^{-x}) = 2$, so $3e^{x} + e^{-x} = 4$, i.e. $3e^{2x} - 4e^{x} + 1 = 0$, i.e. $(3e^{x}-1)(e^{x}-1) = 0$, and $x = 0$ or $x = -\ln 3$. Check both in the original equation – no squaring is ever needed.

Inverse hyperbolic functions

The inverses exist as single functions because sinh and tanh are one-to-one and cosh is one-to-one for $x \geq 0$; their graphs are reflections in $y = x$ of the restricted originals. The **logarithmic equivalents** must be derivable, not just quoted:

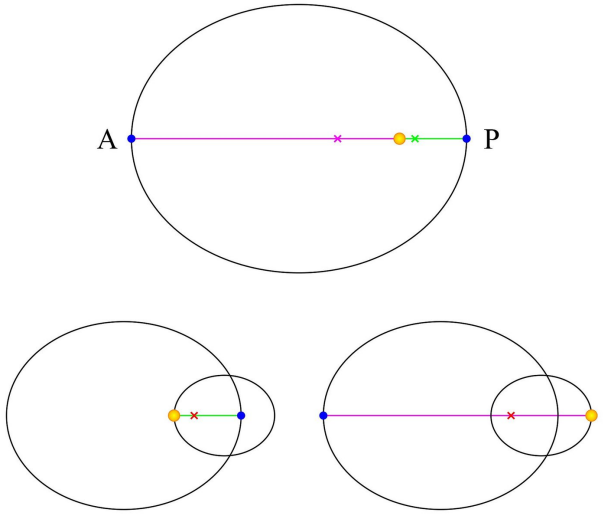
$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right), \quad \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)\ (x \geq 1), \quad \operatorname{artanh} x = \tfrac{1}{2}\ln\tfrac{1+x}{1-x}\ (|x| < 1).$$

Derivation pattern (arsinh). Set $w = \sinh x = \frac{1}{2}(e^{x}-e^{-x})$: then $e^{2x} - 2we^{x} - 1 = 0$, so $e^{x} = w + \sqrt{w^2+1}$ (the negative root is rejected because $e^{x} > 0$) and $x = \ln(w + \sqrt{w^2+1})$. The same quadratic-in-$e^{x}$ structure solves every mixed hyperbolic equation exactly.

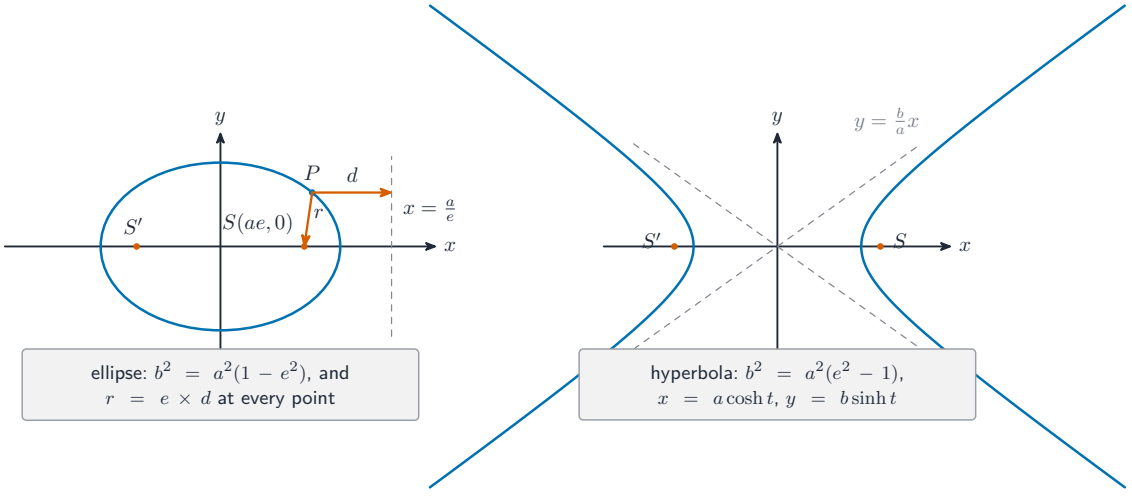
The ellipse and the hyperbola

The **ellipse** 椭圆 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has parametric form $x = a\cos t$, $y = b\sin t$; the **hyperbola** 双曲线 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has $x = a\sec t$, $y = b\tan t$ and also $x = a\cosh t$, $y = b\sinh t$ (the right branch). The **eccentricity** 离心率 e ties everything to the **focus** 焦点 –**directrix** 准线 property $PS = e\cdot PN$ (distance to focus $= e \times$ distance to directrix): for the non-circular ellipse with $a > b > 0$, $b^2 = a^2(1-e^2)$, foci $(\pm ae, 0)$, directrices $x = \pm\dfrac{a}{e}$; for the hyperbola $b^2 = a^2(e^2-1)$ with the same focus and directrix formulas.

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Planets and satellites move in ellipses, with the attracting body at one focus

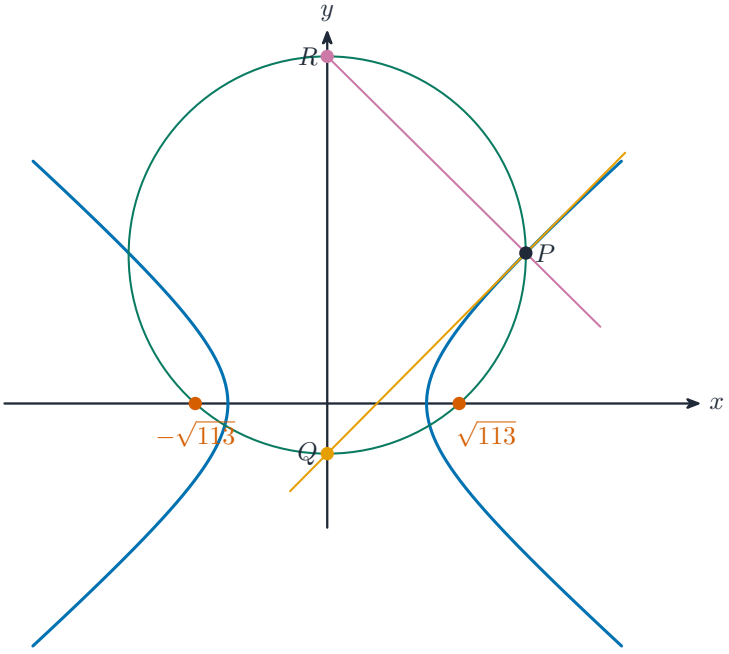


Left: the ellipse with foci $(\pm ae,\,0)$, the directrix $x = a/e$, and the ratio property $PS = e \cdot PN$ marked at a point P. Right: the hyperbola with both branches, asymptotes $y = \pm(b/a)x$ and foci.

> **Worked example – eccentricity and foci (WFM03 June 2025 Q9a).** For H: $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1$: $a^2 = 64$, $b^2 = 49$, so $49 = 64(e^2 - 1)$, $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$; the foci are $(\pm\sqrt{113}, 0)$.

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Substituting $y = mx+c$ into the conic and forcing a zero discriminant gives the tangent condition: $c^2 = a^2m^2+b^2$ for the ellipse, $c^2 = a^2m^2-b^2$ for the hyperbola (so $|m| > \frac{b}{a}$ is needed there). At a parametric point, differentiate instead – no discriminant required. Loci problems replay FP1: distances to foci, midpoints, or a fixed ratio.



tangent meets the y-axis at Q;
normal meets it at R;
$qr = -113$ for every θ, so the circle on QR passes through both foci $(\pm\sqrt{113}, 0)$

drawn for $\theta = \pi/3$

The theta cancels: both foci lie on the circle

Worked example – tangent, normal and a focal circle (WFM03 June 2025 Q9b–e). At $P(8\sec\theta, 7\tan\theta)$ on H: $\dfrac{dy}{dx} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta} = \dfrac{7\sec\theta}{8\tan\theta}$, so the tangent is

$$\tfrac{x}{8}\sec\theta - \tfrac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta = 1,$$

meeting the y-axis at $Q = (0, -7\cot\theta)$; the given normal $8x\cos\theta + 7y\cot\theta = 113$ meets it at $R = (0, \frac{113}{7}\tan\theta)$. The circle on diameter QR (endpoints q, r on the y-axis) has equation $x^2 + (y - \frac{q+r}{2})^2 = (\frac{r-q}{2})^2$, which contains a point $(x, 0)$ exactly when $x^2 = -qr$. Here $qr = -7\cot\theta \cdot \frac{113}{7}\tan\theta = -113$ for every θ, and $113 = (\sqrt{113})^2$ is exactly the squared focal distance – so both foci $(\pm\sqrt{113}, 0)$ lie on the circle. The θ cancelling is the point of the question.

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Differentiating hyperbolic and inverse functions

$\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$, and from the inverses: $\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{x^2+1}}$, $\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2}$ – each provable by inverting $y = f(x)$ and differentiating implicitly, and each valid only on its domain.

Worked example – inverse derivatives on a restricted domain (WFM03 June 2025 Q4). For $y = \operatorname{arsinh} x + \operatorname{arsinh}\frac{1}{x}$ with $x > 0$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}} = \frac{x-1}{x\sqrt{1+x^2}},$$

where $x > 0$ turns $\sqrt{x^2}$ into x – the domain condition is doing real work. Setting $\frac{dy}{dx} = 0$ gives $x = 1$ and $y = 2\operatorname{arsinh} 1 = 2\ln(1+\sqrt{2})$.

Integrating hyperbolic and inverse functions

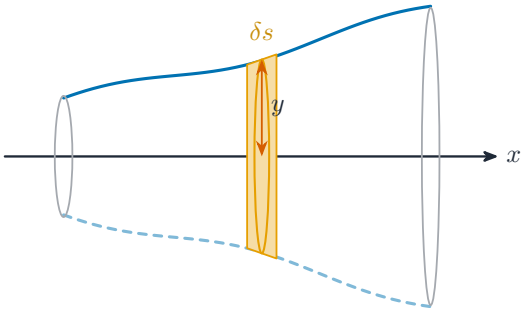
The standard results mirror the derivatives. Complete the square first for surd denominators, and handle inverse functions by parts – take u as the inverse function and 1 as the rest:

Worked example – both techniques (WFM03 June 2025 Q2). (i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$: since $4x^2+8x+9 = 4\left((x+1)^2+\frac{5}{4}\right)$, the integral is $\frac{1}{2}\displaystyle\int \frac{dx}{\sqrt{(x+1)^2+5/4}} = \frac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$ (differentiate back to check). (ii) $\displaystyle\int \operatorname{arcosh} 3x\,dx = x\operatorname{arcosh} 3x - \int \frac{3x}{\sqrt{9x^2-1}}dx = x\operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2-1} + c$.

Substitutions, reduction formulae and surfaces of revolution

Take $a > 0$ below; the displayed inverse-cosh antiderivative uses the branch $x > a$. Four integrals to know cold, each via $x = a\sinh t$ (or $x = a\tan t$, $a\coth$-type splits): $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$. A **reduction formula** 递推公式 expresses I_n through I_{n-1} (or I_{n-2}): integrate by parts with the surd as v, then rewrite the leftover surd integral using the definition of I_n. **Arc length** 弧长 and the **surface of revolution** 旋转曲面 need $s = \displaystyle\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ and $S = 2\pi\displaystyle\int |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (parametrically, $s = \int \sqrt{\dot{x}^2+\dot{y}^2}\,dt$ with $y\sqrt{\dot{x}^2+\dot{y}^2}$ for S).

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band area $\approx 2\pi y\,\delta s$

$$S = 2\pi \int |y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

A surface built from thin bands

Worked example – building a reduction formula (WFM03 June 2025 Q5). With $I_n = \displaystyle\int_1^6 \frac{x^n}{\sqrt{3x-2}}dx$, take $u = x^n$ and $v = \frac{2}{3}\sqrt{3x-2}$:

$$I_n = \left[\tfrac{2}{3}x^n\sqrt{3x-2}\right]_1^6 - \tfrac{2n}{3}\int_1^6 x^{n-1}\sqrt{3x-2}\,dx = \tfrac{8\cdot 6^n - 2}{3} - \tfrac{2n}{3}\left(3I_n - 2I_{n-1}\right),$$

using $\sqrt{3x-2} = \dfrac{3x-2}{\sqrt{3x-2}}$ to expose I_n and I_{n-1}. Collecting: $(3+6n)I_n = 4nI_{n-1} + 8\cdot 6^n - 2$. Seed with $I_0 = \frac{2}{3}(4-1) = 2$ and $I_1 = 6$ (by direct substitution), then $15I_2 = 8(6) + 8(36) - 2 = 334$, and $21I_3 = 12\cdot\frac{334}{15} + 8(216) - 2$, giving $I_3 = \frac{3322}{35}$.

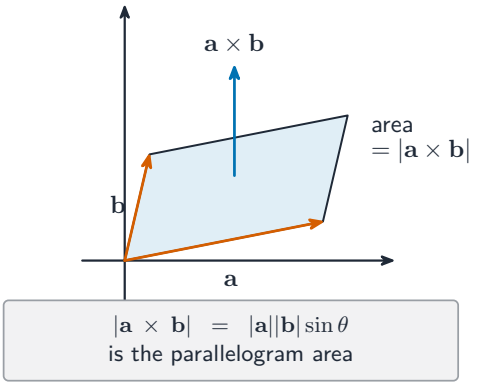
Worked example – a surface with a sinh substitution (WFM03 June 2025 Q7). Rotating $y = \cos 2x$, $0 \le x \le \frac{\pi}{4}$, through 2π about the x-axis (calculator-banned): $S = 2\pi\displaystyle\int_0^{\pi/4} \cos 2x\sqrt{1 + 4\sin^2 2x}\,dx$. With $2\sin 2x = \sinh\theta$, so $4\cos 2x\,dx = \cosh\theta\,d\theta$, the integral becomes $\frac{\pi}{2}\displaystyle\int \cosh^2\theta\,d\theta = \frac{\pi}{4}\Big[\sinh\theta\cosh\theta + \theta\Big]$ between 0 and $\operatorname{arsinh} 2 = \ln(2+\sqrt{5})$, so $S = \frac{\pi}{4}\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$ – the $\cos 2x$ factors cancel exactly, which is why the substitution is supplied.

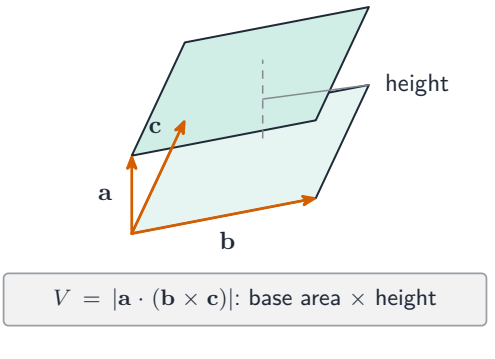
Vectors: products, lines and planes

The **vector product** 向量积 $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is perpendicular to both factors with $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ – the parallelogram area. The **scalar triple product** 混合积 $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is the signed parallelepiped volume; zero means the three vectors are coplanar. Lines: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ or $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = \mathbf{0}$; planes: $\mathbf{r}\cdot\mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ with $\mathbf{n} = \mathbf{b}\times\mathbf{c}$. Distance from a point to a plane: $\dfrac{|\mathbf{x}\cdot\mathbf{n} - p|}{|\mathbf{n}|}$; shortest distance between **skew lines** 异面直线: project the offset between anchor points onto the common

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normal $\mathbf{b}_1 \times \mathbf{b}_2$.





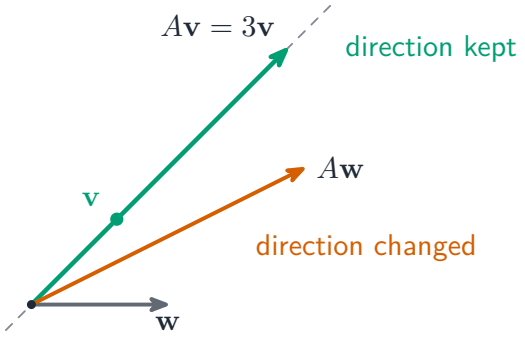
Left: the vector product as the parallelogram area with its normal. Right: the scalar triple product as the volume of the parallelepiped with the height marked.

Worked example – planes to lines to distance (WFM03 June 2025 Q8). For $\Pi_1 : \mathbf{r} = (4,2,-1) + \lambda(7,5,-6) + \mu(-3,1,2)$: the normal is $(7,5,-6) \times (-3,1,2) = (16,4,22) = 2(8,2,11)$, and substituting the anchor point gives $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = 32 + 4 - 11 = 25$. Intersecting with $\Pi_2 : x - y + z = 7$: the line direction is $(8,2,11) \times (1,-1,1) = (13,3,-10)$ and a point solving both planes is $(\frac{13}{5}, -\frac{17}{5}, 1)$, so $(\mathbf{r} - (\frac{13}{5}, -\frac{17}{5}, 1)) \times (13,3,-10) = \mathbf{0}$. The line through $A(2,1,3)$ and $B(3,0,2)$ has direction $(1,-1,-1)$; the common normal is $(13,3,-10) \times (1,-1,-1) = (-13,3,-16)$, and the shortest distance is

$$\frac{\left|(-\frac{3}{5}, \frac{22}{5}, 2) \cdot (-13,3,-16)\right|}{\sqrt{434}} = \frac{\left|\frac{39}{5} + \frac{66}{5} - 32\right|}{\sqrt{434}} = \frac{11}{\sqrt{434}}.$$

Further matrix algebra

Determinants and inverses extend to 3×3: expand along any row or column (signs $+ - + - + -$); $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}(\text{adjugate})$, the adjugate being the transpose of the cofactor matrix. **Singular** 奇异 matrices ($\det = 0$) have no inverse and collapse dimension. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. An **eigenvalue** 特征值 λ with **eigenvector** 特征向量 $\mathbf{v}$ satisfies $\mathbf{Av} = \lambda\mathbf{v}$ – row by row, each component of $\mathbf{Av}$ equals λ times the corresponding component of $\mathbf{v}$ (not λ itself). Given $\mathbf{v} \neq \mathbf{0}$, use any nonzero component v_i and calculate $\lambda = (\mathbf{Av})_i / v_i$, including $\lambda = 0$; otherwise solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and back-substitute. For a symmetric $\mathbf{A}$, eigenvectors for distinct eigenvalues are orthogonal, and an orthogonal matrix $\mathbf{P}$ of normalised eigenvectors **diagonalises** 对角化 it: $\mathbf{P}^T\mathbf{AP} = \mathbf{D}$.



An eigenvector keeps its direction under the matrix; a general vector is turned

Worked example – eigenvalues from a given eigenvector (WFM03 June 2025 Q3). $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ has eigenvector $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Row 1: $1-2+4 = 3 = 3(1)$, so $\lambda = 3$. Row 2: $3 + 2a + b$ must equal $\lambda \times 2 = 6$, and row 3: $a + 2 + b$ must equal $\lambda \times 1 = 3$. So $2a + b = 3$ and $a + b = 1$: subtracting, $a = 2$ and $b = -1$. With $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$: the trace is 2 and, expanding along row 1, $\det \mathbf{M} = 1(-2 + 1) + 1(-3 + 2) + 4(3 - 4) = -1 - 1 - 4 = -6$. The remaining eigenvalues then sum to $2 - 3 = -1$ and multiply to $\frac{-6}{3} = -2$: they are 1 and -2. Eigenvectors: for $\lambda = 1$, solving $(\mathbf{M}-\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $-y+4z = 0$ and $3x+y-z = 0$, so $\mathbf{v} = (-1, 4, 1)$; for $\lambda = -2$, $(\mathbf{M}+2\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $3x-y+4z = 0$ and $2x+y+z = 0$, so $\mathbf{v} = (-1, 1, 1)$ (both row-checked). Trace and determinant are the free verification: $3 + 1 - 2 = 2$ ✓ and $3 \cdot 1 \cdot (-2) = -6$ ✓.

Worked example – singular and inverse with a parameter (WFM03 June 2025 Q6). $\mathbf{A} = \begin{pmatrix} 1 & k & 2 \\ 5 & 3 & -2 \\ 6 & -1 & 4 \end{pmatrix}$: $\det \mathbf{A} = 1(12 - 2) - k(20 + 12) + 2(-5 - 18) = -36-32k$, so $\mathbf{A}$ is singular when $k = -\frac{9}{8}$. The adjugate (transposed cofactors) gives

$$\mathbf{A}^{-1} = \frac{1}{-36 - 32k} \begin{pmatrix} 10 & -4k - 2 & -2k - 6 \\ -32 & -8 & 12 \\ -23 & 6k + 1 & 3 - 5k \end{pmatrix}.$$

Sanity-check the (1,1) entry of the product: $(1, k, 2) \cdot (10, -32, -23) = 10-32k-46 = -36 - 32k$, which cancels against the determinant to give 1 ✓.

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Checklist before you call this unit done

- Define $\cosh, \sinh, \tanh$ from exponentials; derive the core identities; solve mixed equations via e^x quadratics with exact ln answers.
- Derive the $\operatorname{arsinh}, \operatorname{arcosh}, \operatorname{artanh}$ log forms; differentiate inverses with domains respected.
- Use ellipse and hyperbola parametrics; convert between $b^2 = a^2(1-e^2)$ and $a^2(e^2-1)$; place foci and directrices; find tangents/normals at parametric points and via the $y = mx + c$ condition $c^2 = a^2m^2 \pm b^2$.
- Differentiate and integrate hyperbolic and inverse functions, including by parts for $\int$ arcosh-type integrands.
- Know the four surd/rational standard integrals; build reduction formulae by parts and iterate; compute arc lengths and surfaces of revolution exactly.
- Compute $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ with their area and volume meanings; move between plane and line forms; find point-to-plane and skew-line distances.
- Evaluate 3×3 determinants, invert via the adjugate, handle the singular case, find eigenvalues/eigenvectors (normalised when asked) and diagonalise symmetric matrices with an orthogonal $\mathbf{P}$.

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Mechanics 1

Pearson Edexcel · International A-Level · Further Mathematics

Mechanics 1: models, motion and forces

Mechanics 1 (M1, paper WME01/01, 1 hour 30 minutes, 75 marks, first assessment June 2019) is the applied unit of International AS Mathematics: P1 and P2 are assumed. Take $g = 9.8\ \mathrm{m\,s^{-2}}$ unless told otherwise, and give inexact answers to 2 or 3 significant figures. This reference covers modelling assumptions, vectors in mechanics, constant-acceleration kinematics, Newton's laws with connected particles, momentum and impulse, friction, statics and moments.

Must-know for M1 (not in the formula booklet – memorise):

Family	Formulae
momentum	momentum $= mv$, impulse $= mv - mu$
suvat	$v = u + at$, $s = ut + \frac{1}{2}at^2$, $s = vt - \frac{1}{2}at^2$, $v^2 = u^2 + 2as$, $s = \frac{1}{2}(u + v)t$
forces	resultant $= ma$ (component-wise), weight $= mg$, friction $F = \mu R$ when moving
vectors	$\lvert a\mathbf{i} + b\mathbf{j}\rvert = \sqrt{a^2 + b^2}$, bearing from atan in the correct quadrant

Mathematical models in mechanics

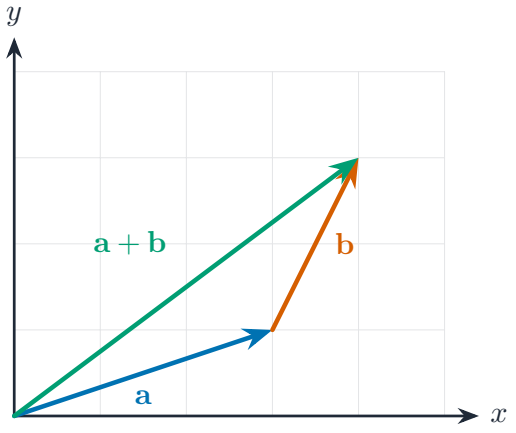
The words in a stem are instructions. A **particle** 粒子 has mass but no size; a **lamina** 薄片 has area but no thickness; a **rod** 杆 has length but no thickness; a **rigid body** 刚体 keeps its shape. **Light** means massless; an unloaded light string over a smooth pulley has the same tension throughout. **Uniform** means constant density, so a uniform rod has its centre of mass at its midpoint. An **inextensible string** 不可伸长绳 keeps its length fixed; a taut string over one fixed pulley links the endpoint speed magnitudes, and a **smooth surface** 光滑表面 versus a **rough surface** 粗糙表面 decide whether friction acts. A **light smooth pulley** 滑轮 changes a tension's direction, not its size. State the assumption you use – "smooth, so no friction" – because the marks are in the modelling as much as the algebra.

Vectors in mechanics

Displacements, velocities, accelerations and forces are vectors: $(3\mathbf{i} - 4\mathbf{j})$ has magnitude 5 and direction $\tan^{-1}(-4/3)$ from the positive x-axis. Resolve into components, work component-wise, and combine at the end. The **resultant** 合力 of $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ is

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$\begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix}$. With **i** east and **j** north, a **bearing** 方位角 is measured clockwise from north: the vector $-5\mathbf{i}-12\mathbf{j}$ points south-west, at $\tan^{-1}(5/12) = 22.6°$ west of due south, so its bearing is $180+23 = 203°$.



Adding by the triangle law: draw **b** *from the tip of* **a**, *and* **a** + **b** *runs from start to finish.*

Worked example – position vectors and bearings (WME01 June 2025 Q2). Particle A has $\mathbf{r}_A(2) = (-5\mathbf{i}+16\mathbf{j})$ and $\mathbf{r}_A(5) = (10\mathbf{i}+4\mathbf{j})$. Velocity $= \dfrac{\Delta\mathbf{r}}{\Delta t} = \dfrac{15\mathbf{i}-12\mathbf{j}}{3} = 5\mathbf{i}-4\mathbf{j}$, so $\mathbf{r}_A(t) = (-5\mathbf{i}+16\mathbf{j})+(t-2)(5\mathbf{i}-4\mathbf{j}) = (5t-15)\mathbf{i}+(24-4t)\mathbf{j}$. With B at $\mathbf{s}_B = -5\mathbf{i}+7\mathbf{j}+t(2\mathbf{i}-3\mathbf{j})$, at $t=5$: $\overrightarrow{AB} = (5\mathbf{i}-8\mathbf{j})-(10\mathbf{i}+4\mathbf{j}) = -5\mathbf{i}-12\mathbf{j}$, bearing 203°.

Kinematics of a particle in a straight line

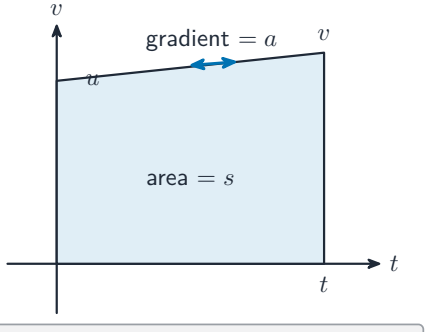
Constant acceleration means the suvat formulae apply once per phase; the discipline is signs. Choose positive, write s, u, v, a, t with signs for each phase, and pick the formula that avoids the unknown you do not want. Graphs: on a velocity–time graph the gradient is acceleration and the area is displacement (area below the axis counts negative); a speed–time graph cannot go negative, so upward motion then downward motion appears as a V.

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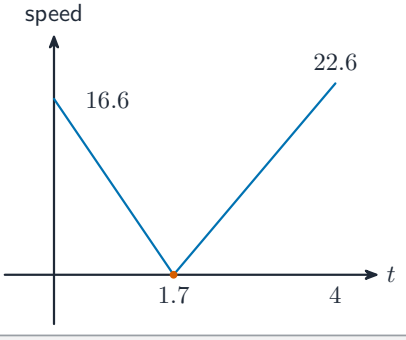


A high-speed train: constant acceleration in a straight line is modelled with the suvat equations

Worked example – up and down (WME01 June 2025 Q3). A ball is projected vertically up from A, 12 m above the ground at B, and takes 4 s to reach B. Taking up positive with A as origin, $s = -12$: $-12 = 4U - 4.9(16)$, so $U = 16.6\ \mathrm{m\,s^{-1}}$ (2 s.f. 17). Speed at the ground: $v^2 = 16.6^2 + 2(9.8)(12) = 510.8$, so $v = 22.6\ \mathrm{m\,s^{-1}}$. The speed–time graph is a V: falling from 16.6 to zero at the top (after $16.6/9.8 = 1.7$ s), then rising to 22.6 at $t = 4$ – the downward leg is longer and climbs to a higher speed; both legs have gradient of size g.



velocity–time: gradient accelerates, area displaces



speed–time V: vertex on the axis, downward leg longer

Left: a velocity-time trapezium with gradient a and area s marked. Right: the speed-time V of the worked example, vertex on the time axis, labelled 16.6, 22.6 and t = 4.

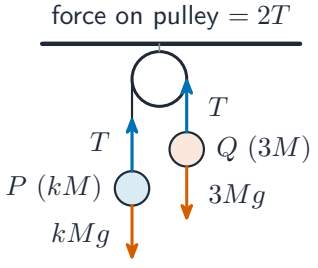
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Newton's laws and connected particles

Newton's second law, applied one way along a line: $F_{\text{net}} = ma$ with every force resolved and signed. For two **connected particles** on a light inextensible string, write an equation of motion for EACH body (same tension, same speed magnitude), then eliminate T. On an inclined plane resolve along and perpendicular to the slope; $\tan\theta = 3/4$ means $\sin\theta = 3/5$, $\cos\theta = 4/5$.

Worked example – two ropes on a barge (WME01 June 2025 Q4). Forces of 800 N at 25° above the line of motion and 500 N at α below it pull a barge straight along the canal. Perpendicular to the motion there is no acceleration: $800\sin 25° = 500\sin\alpha$, so $\sin\alpha = 0.676$ and $\alpha = 43°$. Along the motion, with mass 15000 kg and resistance 750 N: $800\cos 25° + 500\cos\alpha - 750 = 15000a$, giving $a = 0.023\ \text{m s}^{-2}$ – three-decimal answers are expected when the numbers are this small.

Worked example – pulley, start to finish (WME01 June 2025 Q8). Balls P (mass kM) and Q (mass $3M$, $k < 3$) hang over a fixed light smooth pulley and accelerate at $\frac{g}{5}$. Equations of motion: $T - kMg = kM\cdot\frac{g}{5}$ (P up) and $3Mg - T = 3M\cdot\frac{g}{5}$ (Q down). Eliminating T: $3Mg - kMg = \frac{g}{5}(3M + kM)$, so $5(3-k) = 3 + k$ and $k = 2$. The string pulls the pulley down with $2T$: with $M = 0.5$ kg, $T = 1(9.8 + 1.96) = 11.76$ N, so the force is 23.5 N. When Q lands after falling 2.5 m its speed is $\sqrt{2\cdot\frac{g}{5}\cdot 2.5} = \sqrt{g} = 3.13$; if it rebounds at $0.4\ \text{m s}^{-1}$, the impulse on Q is $1.5(3.13 + 0.4) = 5.3$ N s. P then coasts upward against gravity alone for $\frac{\sqrt{g}}{g} = 0.32$ s after a $5/\sqrt{g} = 1.60$ s descent phase: total time to rest ≈ 1.9 s.



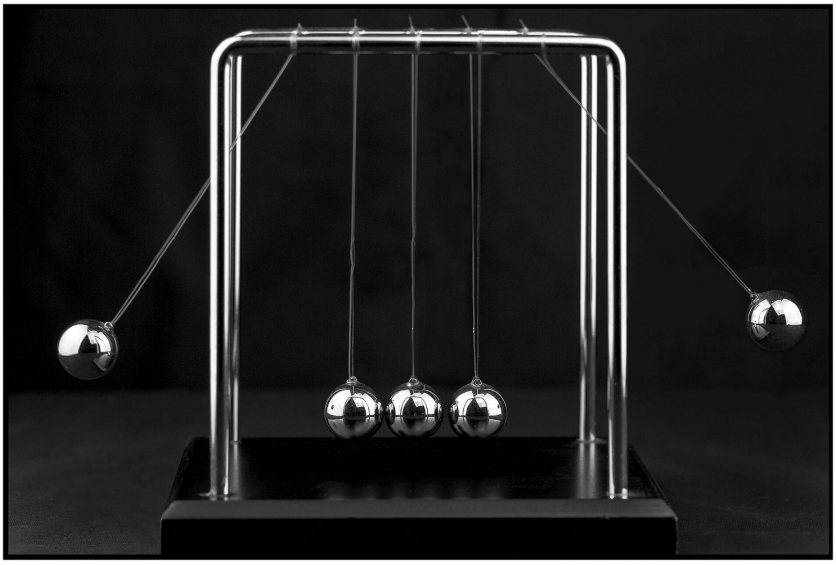
P accelerates up and Q down at $\frac{g}{5}$; same string, same tension: $T - kMg = \frac{kMg}{5}$ and $3Mg - T = \frac{3Mg}{5}$

A light smooth pulley with P of mass kM on the left and Q of mass 3M on the right, tensions T up each side, weights drawn, and the force 2T on the pulley marked.

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Momentum and impulse

Momentum mv and **impulse** 冲量 $I = mv - mu$ are signed along the line. Newton's third law makes the impulse exerted on each of two colliding bodies equal in magnitude and opposite in direction – use one of them, not both. For a direct collision, conservation of momentum $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ needs consistent signs; restitution is NOT in M1.



A Newton's cradle: momentum passes along the line of balls in each collision

> **Worked example – impulse on both bodies (WME01 June 2025 Q1).** P (mass km, speed $5u$) catches Q (mass $3m$, speed $2u$); afterwards P still moves forward at $2u$ and Q at v. The impulse on Q is $4.5mu$: $3m(v - 2u) = 4.5mu$, so $v = 3.5u$. The impulse on P is equal and opposite: $km(5u - 2u) = 4.5mu$, so $k = 1.5$.

Friction

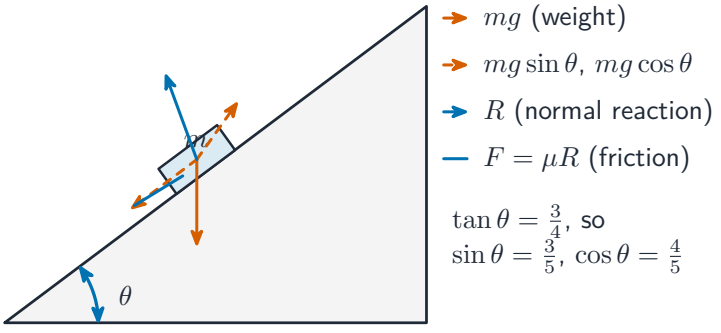
On a **rough** surface the contact force splits into the **normal reaction** 法向反力 R (perpendicular) and friction F (along, opposing motion or its tendency). Moving: $F = \mu R$ with μ the **coefficient of friction** 摩擦系数. In equilibrium $F \leq \mu R$, with equality at limiting equilibrium – the phrase "on the point of moving" licenses $F = \mu R$.

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A climber relies on friction: the limiting value is μR, so steeper rock needs more normal reaction

Worked example – a package on a rough slope (WME01 June 2025 Q6). With $\tan\theta = \frac{3}{4}$ and $\mu = \frac{1}{4}$: perpendicular, $R = mg\cos\theta = \frac{4}{5}mg$; along (up positive, decelerating): $-mg\sin\theta - \frac{1}{4}R = ma$, so $a = -g(\frac{3}{5} + \frac{1}{5}) = -\frac{4}{5}g$ – the deceleration is $\frac{4}{5}g$, and the answer must stay as an exact fraction of g. Held at rest on the slope by a force X up the plane (mass 2 kg): $X + \frac{1}{4}(2g \cdot \frac{4}{5}) = 2g \cdot \frac{3}{5}$, so $X = \frac{4}{5}g = 7.84$ N.



resolve both ways: $R = \frac{4}{5}mg$ and $a = g(\sin\theta + \mu\cos\theta) = \frac{4}{5}g$ when $\mu = \frac{1}{4}$

A block of mass m on a plane inclined at theta with tan theta = 3/4, showing the weight resolved into mg sin theta and mg cos theta, the normal reaction R, and friction F = mu R acting down the slope.

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Statics of a particle

A particle is in **equilibrium** 平衡 when the resultant force is zero: resolve in two perpendicular directions and set both totals to zero, or draw a closed vector triangle. Weight mg down, **normal reaction** perpendicular to the surface, **tension** 张力 along a string away from the body (thrust along a rod pushes), friction along the surface opposing slipping. On rough ground at rest, F is whatever closes the polygon, capped at μR.

> **Worked example – forces in equilibrium (original).** A particle of weight 50 N hangs from two light strings at 40° and 60° to the vertical. Resolving horizontally: $T_1 \sin 40° = T_2 \sin 60°$; vertically: $T_1 \cos 40° + T_2 \cos 60° = 50$. From the first, $T_2 = 0.742 T_1$; then $T_1(0.766 + 0.371) = 50$, so $T_1 = 44$ N and $T_2 = 33$ N (2 s.f.).

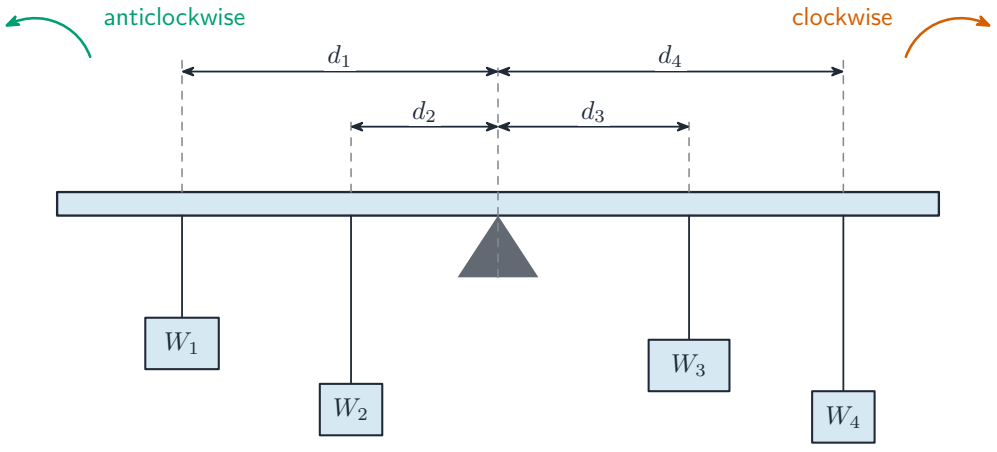
Moments

The **moment of a force** 力矩 about a point is force × perpendicular distance, with a sign per sense. A rigid body in equilibrium has zero resultant moment about ANY point – so pick the pivot that kills the most unknowns. "On the point of tilting about X" means the reaction elsewhere has dropped to zero and moments are taken about X.



A tower crane: the counterweight's moment balances the load's moment about the mast

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Weights on a rule balanced at a pivot – used to test the principle of moments

Worked example – a rod on a ledge (WME01 June 2025 Q5). A non-uniform rod AB of length $3a$ rests on a horizontal ledge with the edge at C, $AC = 2a$, centre of mass d from A. An upward force P at B tilts it about A: moments about A give $3aP = Wd$. A downward force $1.25P$ at B tilts it about C: $1.25Pa = W(2a - d)$. Substituting $P = \frac{Wd}{3a}$ (in units of a): $1.25 \cdot \frac{d}{3} = 2 - d$, so $1.25d = 6a - 3d$, giving $4.25d = 6a$ and $d = \frac{24a}{17}$. The lesson: each tilting case has its own pivot, and only the weight's moment survives unchanged between them.

Checklist before you call this unit done

- Name the modelling assumptions in play (particle, light rod, inextensible string, smooth/rough) and say what each buys you.
- Work with position and velocity vectors component-wise; convert directions to bearings; find relative positions at a common time.
- Solve one-phase and multi-phase suvat problems with signs fixed per phase; read gradients and areas off motion graphs; sketch the speed-time V for up-then-down motion.
- Apply $F = ma$ per body along the line of motion; handle connected particles over pulleys with paired equations; find the force on a pulley as $2T$.
- Use impulse-momentum on ONE body of a collision pair; conserve momentum with consistent signs.
- Split contact forces into R and F; apply $F = \mu R$ when moving or limiting; resolve on inclined planes using $\sin\theta = 3/5$, $\cos\theta = 4/5$ for $\tan\theta = 3/4$.
- Close force polygons for equilibrium in two directions; cap static friction at μR.
- Take moments about the pivot that removes unknowns; switch pivots when the tilting point changes.

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Mechanics 2

Pearson Edexcel · International A-Level · Further Mathematics

Mechanics 2: projectiles, centres of mass, energy and collisions

Mechanics 2 (M2, paper WME02/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the IAL applied unit that builds on M1: P1–P4 and M1 are assumed. Take $g = 9.8\ \mathrm{m\,s^{-2}}$ unless the question's model says otherwise (one June 2025 question uses 10). This reference covers variable-acceleration kinematics, centres of mass, work, energy and power, collisions with restitution, and statics of rigid bodies.

Must-know for M2 (not in the booklet):

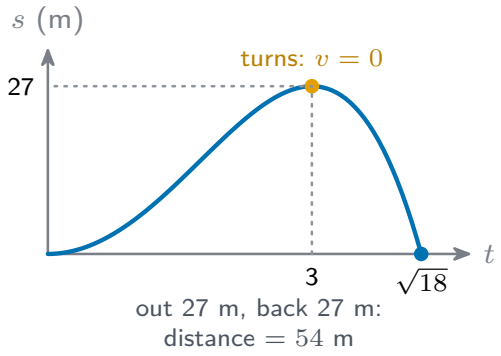
Family	Formulae
energy	**kinetic energy** 动能 $= \frac{1}{2}mv^2$, **potential energy** 势能 $= mgh$, work $= Fs$, power $= Fv = \frac{dW}{dt}$
calculus kinematics	$v = \int a\,dt$, $s = \int v\,dt$ (constants closed by initial conditions)
restitution	$e = \dfrac{v_B - v_A}{u_A - u_B}$ along the line, $0 \le e \le 1$
M1 carryover	momentum mv, impulse $= mv - mu$, $F = \mu R$, moments

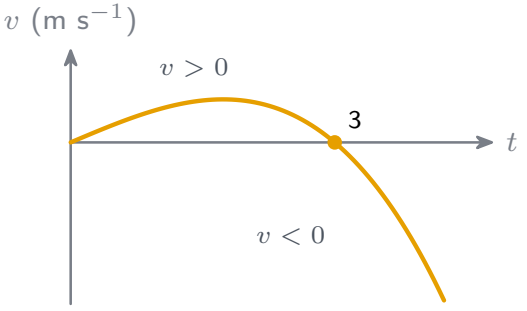
Kinematics with variable acceleration

When acceleration is a function of time, **integrate**: 积分 $v = \int a\,dt$ and $s = \int v\,dt$, with the constants fixed by the initial conditions ("at rest at O" kills both). For total DISTANCE when the particle turns around, find the stopping time ($v = 0$) and add both legs. Position vectors differentiate component-wise: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, $\mathbf{a} = \frac{d\mathbf{v}}{dt}$.

> **Worked example – integration with a trap (WME02 June 2025 Q1).** With $a = k(3-t^2)$ from rest at O and $v(1.5) = 13.5$: $v = k(3t - \frac{t^3}{3})$ (constant zero by rest), and $k(4.5 - 1.125) = 3.375k = 13.5$ gives $k = 4$. Then $s = 6t^2 - \frac{t^4}{3} = \frac{t^2}{3}(18 - t^2)$. For the total distance back to O: $v = \frac{4t}{3}(9 - t^2) = 0$ at $t = 3$, $s(3) = 27$, so the particle travels $27 + 27 = 54$ m – the return leg is a second 27, not a subtraction.

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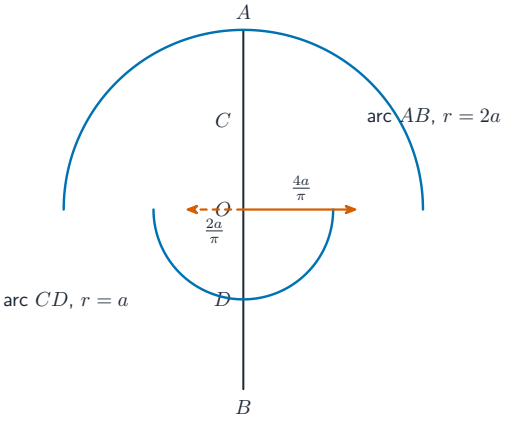




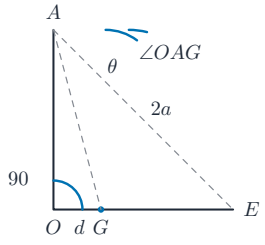
The particle stops at $t = 3$, 27 m from O, then comes back: the total distance is $27 + 27 = 54$ m, while the displacement at return is 0

> **Worked example – vector velocity then impulse (WME02 June 2025 Q2).** With $\mathbf{r} = 4\sqrt{t}\,\mathbf{i} - 3t\mathbf{j}$ (metres, t seconds), mass 3 kg: $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 2t^{-1/2}\mathbf{i} - 3\mathbf{j}$, so at $t = 4$ the velocity is $\mathbf{i} - 3\mathbf{j}$. An impulse $(6\mathbf{i} - 3\mathbf{j})$ N s gives $3(\mathbf{v}' - (\mathbf{i} - 3\mathbf{j})) = (6\mathbf{i} - 3\mathbf{j})$, so $\mathbf{v}' = 3\mathbf{i} - 4\mathbf{j}$ and the speed is 5 m s^{-1} – differentiate FIRST, apply the impulse at the stated instant second.

Centres of mass: particles, laminas and wire frameworks



signed moments: $2\pi a \cdot \frac{4a}{\pi} - \pi a \cdot \frac{2a}{\pi} = 6a^2$ over $(3\pi + 2)a$



$\tan\angle OAG = \frac{3}{3\pi+2}$ and $\theta = 45 - \angle OAG$

Left: the S-shaped wire framework - straight line A-C-O-D-B with the outer semicircular arc of radius 2a to the right and the inner arc of radius a to the left, COM distances 4a/pi and 2a/pi marked from the line. Right: the construction triangle A-O-E with the right angle at O, G on OE at distance d, the dashed hanging line AG, and the angles theta and OAG at A.

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For particles: the **centre of mass** 质心 is $\bar{x} = \dfrac{\sum mx}{\sum m}$. For uniform bodies use the table method – split into pieces whose COM you know (booklet: semicircular wire $\frac{2r}{\pi}$, semicircular lamina $\frac{4r}{3\pi}$, sector, arc), weight each piece by MASS (lamina: area; wire: length), and take moments about a convenient line. Pieces on opposite sides of the line carry opposite signs. Composite laminas subtract holes as negative areas.

> **Worked example – an S-shaped wire framework 线框架 (WME02 June 2025 Q3).** Two semicircular wire arcs, AB of radius $2a$ and CD of radius a, both centred at O on opposite sides of the straight wire ACDB ($AC = DB = a$). Table (about the line AB):

piece	length	COM from AB	moment
arc AB	$2\pi a$	$\frac{2(2a)}{\pi} = \frac{4a}{\pi}$	$8a^2$
arc CD	πa	$-\frac{2a}{\pi}$ (opposite side)	$-2a^2$
straights	$2a$	0	0

Total length $(3\pi + 2)a$; moments $6a^2$; so the centre of mass is $\dfrac{6a}{3\pi + 2}$ from AB. Suspended from A with E on the arc where $\angle AOE = 90°$: the symmetry line OE carries G, so $\tan\angle OAG = \dfrac{3}{3\pi + 2}$, and since $OA = OE$ makes $\angle OAE = 45°$, the angle θ of AE below the vertical satisfies $\theta = 45° - \angle OAG$, giving $\tan\theta = \dfrac{3\pi - 1}{3\pi + 5}$.

Suspending laminas and frameworks

A freely suspended body hangs with its centre of mass vertically below the point of suspension. The angle asked for is usually between a drawn chord of the body and the vertical: find the angle at the suspension point by right-angled trigonometry with the COM distance, then adjust by any geometry angle (isosceles triangles are the standard source of 45°, 60°).

Work, energy and power

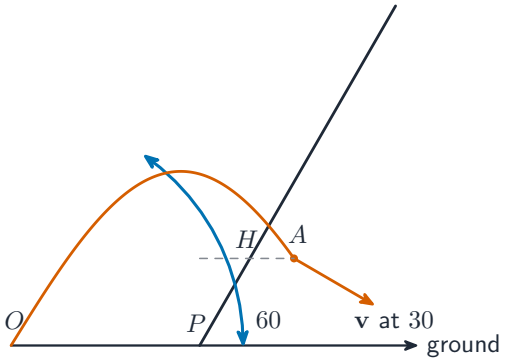
Work done 功 by a constant force: Fs along the motion (negative against it); gravity does $\pm mgh$. The **work-energy principle** 功能原理: total work = change in KE. **conservation of mechanical energy** 机械能守恒: energy is conserved when only gravity does work; resistance enters as work $-Rs$. **power** 功率 $P = Fv$: at constant speed the driving force balances resistance plus gravity-components, so $v = \frac{P}{F}$; at a given speed the excess force accelerates.

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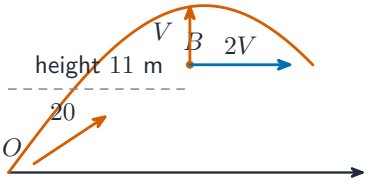


A roller coaster: potential energy at the top becomes kinetic energy lower down, less the work done against resistance

Worked example – a car on a slope (WME02 June 2025 Q4). A 750 kg car climbs a slope with $\sin\alpha = \frac{1}{7}$, resistance 200 N, engine at 20 kW. Constant speed: $\frac{20000}{V} = 750(9.8)\frac{1}{7} + 200 = 1250$, so $V = 16\ \mathrm{m\,s^{-1}}$. At $10\ \mathrm{m\,s^{-1}}$: $2000 - 1250 = 750a$, so $a = 1\ \mathrm{m\,s^{-2}}$. Engine off at A at $10\ \mathrm{m\,s^{-1}}$: work-energy gives $\frac{1}{2}(750)(100) = 750g(d\sin\alpha) + 200d$, i.e. $37500 = 1250d$ and $d = 30$ m – the resistance acts over the SLOPE distance d, gravity over the height $d\sin\alpha$.



perpendicular impact fixes the velocity direction at A



energy: $\frac{1}{2}m(20^2) = \frac{1}{2}m(5V^2) + mg(11)$

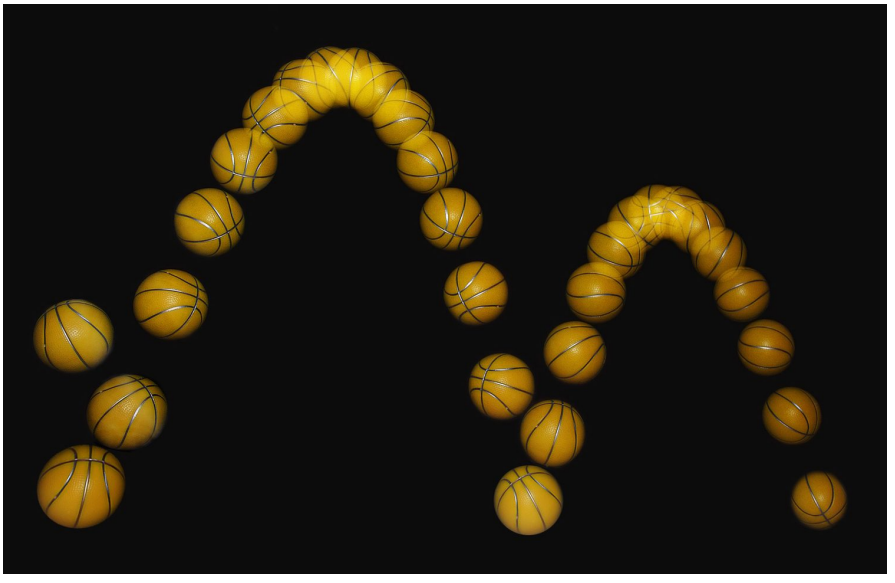
Left: a projectile trajectory meeting a 60-degree plane perpendicularly at A, with the velocity arrow at 30 degrees below the horizontal and the height H marked. Right: the same trajectory with the velocity components V (vertical) and 2V (horizontal) marked at the 11 m point.

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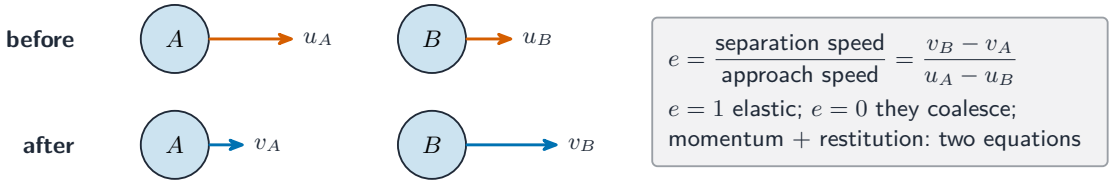
Worked example – a projectile closed by energy (WME02 June 2025 Q7). A **projectile** 抛体 on a parabolic **trajectory** 轨迹 leaves O at 20 $\mathrm{m\,s^{-1}}$ ($g = 10$) is 11 m up when its vertical component is V and its horizontal component $2V$. Energy: $200 = \frac{1}{2}(V^2 + 4V^2) + 110$, so $V = 6$ (components 6 and 12). It meets a plane at 60° to the horizontal PERPENDICULARLY, so the velocity is at 30° below horizontal: speed $= \frac{12}{\cos 30°} = 8\sqrt{3}$, and energy from O gives $200 = \frac{1}{2}(192) + 10H$, so $H = 10.4 \to 10$ m (2 s.f.).

Collisions and restitution

Newton's experimental law, the **coefficient of restitution** 恢复系数: $e = \dfrac{v_B - v_A}{u_A - u_B}$ (separation speed over approach speed, $0 \le e \le 1$; $e = 1$ elastic, $e = 0$ they stick). With conservation of momentum, two equations solve any direct collision. The KE loss always factorises with $(1 - e^2)$: loss $= \frac{m_A m_B}{2(m_A + m_B)}(1 - e^2)(u_A - u_B)^2$ – and the impulse on either body comes from its own momentum change. A rebound from a fixed wall multiplies the speed by e outright.



A bouncing ball: each rebound speed is e times the impact speed, so the bounces shrink



Newton's experimental law: separation speed = e × approach speed

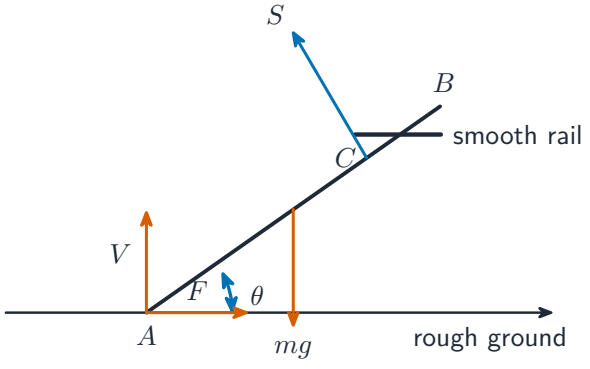
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Worked example – the full collision story (WME02 June 2025 Q5). P (mass $2m$, speed u) strikes Q (mass m) at rest, restitution e. Momentum: $2u = 2v_P + v_Q$; restitution: $eu = v_Q - v_P$. Solving: $v_Q = \frac{2(1+e)u}{3}$, $v_P = \frac{(2-e)u}{3}$. The KE after is $\frac{mu^2}{3}(2+e^2)$, so the loss is $\frac{mu^2}{3}(1-e^2)$ – the factor $k = \frac{1}{3}$. If the impulse on Q is $\frac{8}{9}mu$: $m \cdot \frac{2(1+e)u}{3} = \frac{8mu}{9}$ gives $e = \frac{1}{3}$. Q then rebounds from a wall with restitution f to match P's speed $\frac{5}{9}u$: $f \cdot \frac{8}{9}u = \frac{5}{9}u$, so $f = \frac{5}{8}$.

Successive impacts

Three-body and wall chains are bookkeeping: solve the first collision completely, carry the surviving speeds into the next impact, and only equate quantities the question ties together ("same speed", "at rest"). Wall rebounds reverse direction and scale speed by e; keep a signed diagram per stage.

Statics of rigid bodies



moments about A: $S(1.5a) = mga\cos\theta$,
and $F = S\sin\theta$ at limiting equilibrium

A uniform rod AB leaning at angle theta with end A on rough ground, resting against a smooth horizontal rail at C (AC = 1.5a): the weight mg at the midpoint, the rail reaction S perpendicular to the rod, and the floor components F (horizontal) and V (vertical) at A.

A **rigid body** 刚体 in equilibrium needs zero resultant force AND zero resultant **moment** 力矩 about any point. Choose the pivot to eliminate unknowns (a smooth rail's single unknown reaction is why moments about the floor contact pay off). Resolve horizontally and vertically for the contact components; on a rough floor at **limiting equilibrium** 极限平衡 **friction** 摩擦力 $F = \mu R$ with F horizontal.

Worked example – a rod against a smooth rail (WME02 June 2025 Q6). A uniform rod AB (mass m, length $2a$) rests with A on rough ground against a smooth rail at C, $AC = 1.5a$, inclination θ, the **normal reaction** 法向反力 S of the rail, perpendicular to the rod. Moments about A: $S(1.5a) = mg\,a\cos\theta$, so $S = \frac{2mg\cos\theta}{3}$. Vertically: $V = mg - S\cos\theta = \frac{mg}{3}(3 - 2\cos^2\theta)$. Limiting equilibrium with $\mu = \frac{4}{7}$: $F = S\sin\theta$, so $\frac{4}{7} \cdot \frac{mg}{3}(3 - 2\cos^2\theta) = \frac{2mg}{3}\cos\theta\sin\theta$. Dividing by $\cos^2\theta$ and using $\sec^2\theta = 1 + \tan^2\theta$: $12\tan^2\theta - 14\tan\theta + 4 = 0$, i.e. $6\tan^2\theta - 7\tan\theta + 2 = 0$.

Checklist before you call this unit done

- Integrate $a(t)$ to v and s with constants from the initial conditions; handle total distance via the turning point; differentiate $\mathbf{r}(t)$ for vector velocity and apply impulses as $\mathbf{I} = m\Delta\mathbf{v}$.
- Build COM tables (mass/area/length weighting) with signed distances for opposite-side pieces and negative holes; use the booklet's wire and lamina results; find suspension angles by trigonometry at the suspension point plus any isosceles-triangle angle.
- Apply work-energy and conservation of mechanical energy, with resistance work over slope distance and $P = Fv$ both ways.
- Solve direct collisions with momentum + restitution; state both speeds; compute the $(1 - e^2)$ energy loss and per-body impulses; chain wall rebounds and second collisions with a signed diagram.
- For rigid bodies: moments about the unknown-killing pivot, resolve for components, and close limiting-equilibrium problems into an equation in $\tan\theta$.

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Mechanics 3

Pearson Edexcel · International A-Level · Further Mathematics

Mechanics 3: elastic strings, SHM, circles and centres of mass

Mechanics 3 (M3, paper WME03/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the final mechanics unit: P1–P4, M1 and M2 are assumed. The whole paper works symbolically – answers stay in m, g, a, L, r, and exact algebra is the marks. This reference covers acceleration as a function of x, elastic strings and springs, variable forces and simple harmonic motion, horizontal and vertical circular motion, and centres of mass by integration.

Must-know for M3 (not in the booklet):

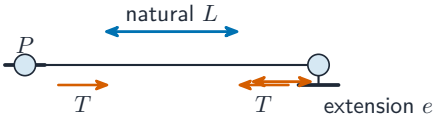
Family	Formulae
Hooke's law	tension $T = \dfrac{\lambda e}{L}$ (λ modulus, e extension, L natural length)
elastic energy	$E = \dfrac{\lambda e^2}{2L}$
SHM	$\ddot{x} = -\omega^2 x$, **period** 周期 $\frac{2\pi}{\omega}$, $v^2 = \omega^2(a^2 - x^2)$
circles	$a = r\omega^2 = \frac{v^2}{r}$ towards the centre; $v = r\omega$
COM	$\bar{x} = \dfrac{\int x\,y^2\,dx}{\int y^2\,dx}$ for a solid of revolution about x
kinematics	$a = v\dfrac{dv}{dx}$ when $a = f(x)$

Kinematics with acceleration as a function of x

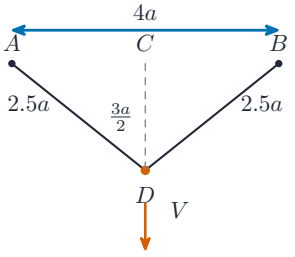
When acceleration depends on displacement, replace $\frac{dv}{dt}$ with $v\dfrac{dv}{dx}$ and integrate against x; when it depends on time, separate $\frac{dx}{dt}$. The initial condition closes each constant – state it before substituting.

> **Worked example (WME03 June 2025 Q3).** With $a = 4x - 2$ and at $t = 1$, $x = 0$, $v = 1$: (a) $v\frac{dv}{dx} = 4x - 2$ gives $\frac{1}{2}v^2 = 2x^2 - 2x + \frac{1}{2}$, so $v^2 = (2x-1)^2$ and (positive direction) $v = 1 - 2x$. (b) Substituting $v = 1 - 2x$ into $a = \frac{dv}{dt}$ gives $\frac{dv}{dt} = -2v$ (or separate $\frac{dx}{dt} = 1 - 2x$): $\ln v = -2t + C$ with $v(1) = 1$ gives $v = e^{2-2t}$.

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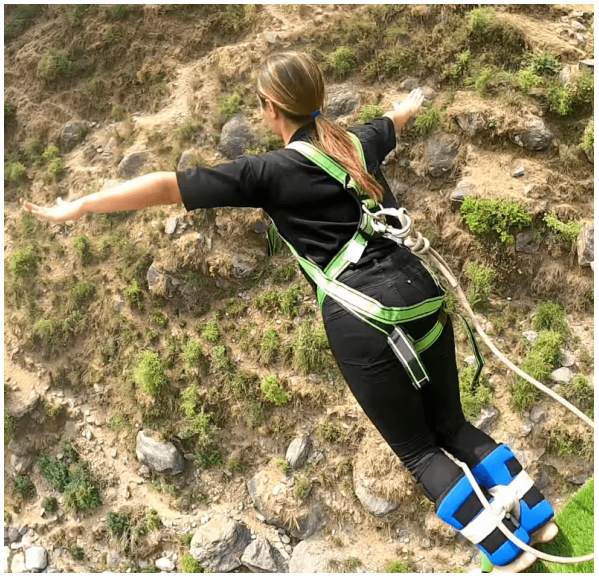


$T = \frac{\lambda e}{L}$, energy $= \frac{\lambda e^2}{2L}$



each half: natural $\frac{3a}{2}$, so at D energy $\frac{1}{2}mga$ total

Left: a light elastic string of natural length L stretched by an extension e, tension T pulling inwards at both ends. Right: the two-part midpoint string with $AB = 4a$ horizontal, P hanging at D with $CD = 3a/2$ and each half-string $2.5a$.



A bungee jump: the cord is slack until it passes its natural length, then Hooke's law applies

Hooke's law 胡克定律: a light elastic string (or spring) of **natural length** 原长 L and **modulus of elasticity** 弹性模量 λ stretched by e exerts the tension $T = \frac{\lambda e}{L}$ – tension pulling inwards at both ends. The **elastic energy** 弹性势能 stored is $\frac{\lambda e^2}{2L}$. **Work–energy principle** 功能原理 problems now carry three energy stores: kinetic, gravitational and elastic, plus any work done against resistance.

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Worked example – a midpoint on a two-part string (WME03 June 2025 Q4). A string with $\lambda = mg$, natural length $3a$, fixed between A and B ($AB = 4a$ horizontal); P hangs from the midpoint, projected downward from C (the midpoint of AB) at $\sqrt{3ag/2}$, and reaches D with $CD = \frac{3}{2}a$. Each half of the string has natural length $\frac{3a}{2}$: at C the half-length is $2a$ (extension $\frac{a}{2}$); at D it is $\sqrt{(2a)^2 + (\frac{3}{2}a)^2} = \frac{5a}{2}$ (extension a). Elastic energy gain per half: $\dfrac{mg}{2 \cdot \frac{3a}{2}}\left(a^2 - \frac{a^2}{4}\right) = \dfrac{mg}{3a} \cdot \frac{3a^2}{4} = \frac{mga}{4}$; both halves: $\frac{1}{2}mga$ ✓. Then with resistance $\frac{1}{5}mg$ over $\frac{3a}{2}$ and the loss of gravitational energy $mg \cdot \frac{3a}{2}$:

$$\tfrac{3}{2}mga = \tfrac{1}{2}mV^2 - \tfrac{1}{2}m \cdot \tfrac{3ag}{2} + \tfrac{1}{2}mga + \tfrac{1}{5}mg \cdot \tfrac{3a}{2} \implies V^2 = \tfrac{29}{10}ag,$$

so $V = \sqrt{29ag/10}$.

Worked example – a hinged rod (WME03 June 2025 Q1). A uniform rod AB (mass m, length $2a$) hinged at A; an elastic string ($\lambda = 2mg$, natural length L) runs vertically from B to C, where AC is horizontal and BC $= 1.2a$ – the geometry puts B at $1.6a$ horizontally from A and $1.2a$ below it (a 3–4–5 triangle), so the rod's midpoint is $0.8a$ from the vertical through A. Moments about A: $T \cdot 1.6a = mg \cdot 0.8a$, so $T = \frac{1}{2}mg$; then Hooke's law gives $\frac{1}{2}mg = \frac{2mg}{L}(1.2a - L)$, i.e. $L = 0.96a = \frac{24}{25}a$.

Variable forces and simple harmonic motion

With variable force $F(x)$ in one dimension: $m\dfrac{dv}{dt} = F$ becomes $mv\dfrac{dv}{dx} = F(x)$; inverse-square gravitation appears as $F = \dfrac{k}{x^2}$. **Simple harmonic motion** 简谐运动 (SHM) means $\ddot{x} = -\omega^2 x$ – the sign is the proof. Standard results (quotable): period $\frac{2\pi}{\omega}$, $v^2 = \omega^2(a^2 - x^2)$, $x = a\cos\omega t$ from an end, $a\sin\omega t$ from the centre. A particle between two elastic strings oscillates with SHM whenever the net force is proportional to displacement from the equilibrium point.



A pendulum clock keeps time using simple harmonic motion.

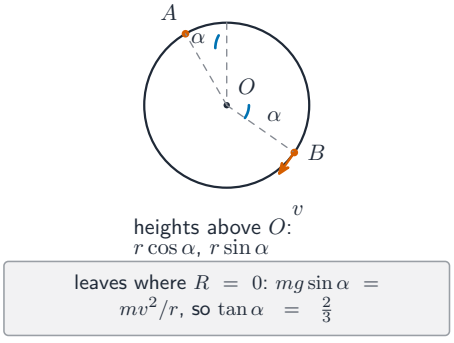
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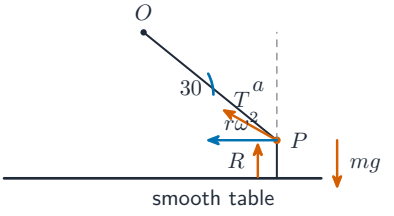
Worked example – SHM between two strings (WME03 June 2025 Q7). Strings from P to A ($\lambda = mg$, natural $3L$) and to B ($\lambda = 2mg$, natural $2L$), $AB = 7L$; equilibrium at E with $AE = 4.5L$ (each tension $= \frac{mg}{2}$ – the check that E is genuine). At displacement x from E towards B, the B-string shortens and the A-string lengthens:

$$m\ddot{x} = \frac{2mg}{2L}\left(\tfrac{L}{2} - x\right) - \frac{mg}{3L}\left(\tfrac{3L}{2} + x\right) = -\frac{4mg}{3L}x,$$

so P moves with SHM, $\omega^2 = \frac{4g}{3L}$ and the period is $\frac{2\pi}{\omega} = \pi\sqrt{\frac{3L}{g}}$. Released from C with $AC = 5L$ with **amplitude** 振幅 $\frac{L}{2}$: $v_{max} = \frac{L}{2}\omega = \sqrt{\frac{gL}{3}}$. The time with $|v| \le \sqrt{\frac{gL}{12}} = \frac{v_{max}}{2}$: with $x = \frac{L}{2}\cos\omega t$ from the end, $v \propto \sin\omega t$ and $|\sin\omega t| \le \frac{1}{2}$ holds for a third of every oscillation, so the slow time is $\frac{1}{3} \times \pi\sqrt{\frac{3L}{g}} = \pi\sqrt{\frac{L}{3g}}$.

Motion in a horizontal circle





$T\cos 30 = m(a\cos 30)\omega^2$; $R = \frac{3}{4}mg$

Left: a sphere of radius r with the particle at A (OA at alpha to the upward vertical) and at B (OB at alpha to the horizontal), the tangent velocity at B and the R = 0 condition. Right: a particle P on a smooth table tethered to O above at 30 degrees, with the centre-seeking and vertical force arrows.

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A swing ride: each seat is a conical pendulum moving in a horizontal circle

Angular speed 角速度 ω (rad s^{-1}) with $v = r\omega$; the **centripetal acceleration** 向心加速度 is $r\omega^2 = \frac{v^2}{r}$, directed towards the centre. Resolve towards the centre for the force equation and vertically for equilibrium – the **conical pendulum** 圆锥摆, a particle on a table tethered by a string, and banked surfaces are the three standard contexts.

> **Worked example – a circle on a table (WME03 June 2025 Q2).** P (mass m) on a smooth table, attached to O above by a string of length a making $30°$ with the table, moving in a circle with $\omega = \sqrt{\frac{g}{2a}}$. The circle's radius is $a\cos 30°$. Towards the centre: $T\cos 30° = m(a\cos 30°)\omega^2 = ma\cos 30° \cdot \frac{g}{2a}$, so $T = \frac{mg}{2}$. Vertically: $R = mg - T\sin 30° = mg - \frac{mg}{4} = \frac{3mg}{4}$.

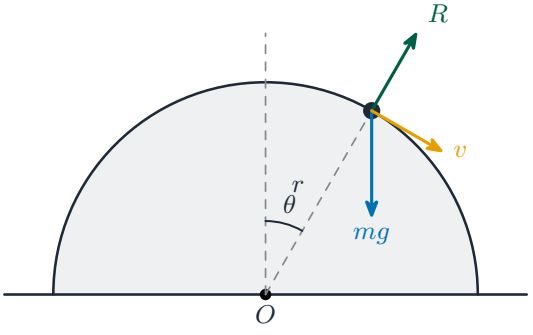
Motion in a vertical circle

A **vertical circle** 竖直圆周 pairs energy with Newton's law towards the centre. The change in kinetic energy equals the loss of gravitational and elastic potential energy when there are no other energy transfers. This energy equation combines with Newton's law towards the centre; the leaving condition on a surface is $R = 0$, on a string $T = 0$. After leaving, the particle is a projectile: the horizontal component of velocity is conserved.

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A vertical loop: speed changes with height, so energy and the radial equation are used together



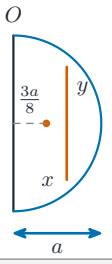
towards O: $mg\cos\theta - R = \dfrac{mv^2}{r}$
energy: $v^2 = 2gr(1-\cos\theta)$ from rest at the top
leaves when $R = 0$, then a projectile

Leaving a smooth sphere: resolve towards the centre, use energy for v^2, and set $R = 0$

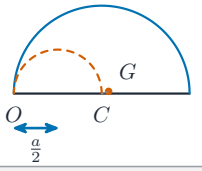
Worked example – leaving a sphere (WME03 June 2025 Q5). P released at A on a smooth sphere (radius r), OA at α above the vertical; it leaves at B, OB at α above the horizontal. (a) Energy A→B: the height of A above O is $r\cos\alpha$ and the height of B above O is $r\sin\alpha$, so P falls $r(\cos\alpha - \sin\alpha)$: $\frac{1}{2}mv^2 = mgr(\cos\alpha - \sin\alpha)$, giving $v^2 = 2gr(\cos\alpha - \sin\alpha)$. (b) At B with $R = 0$: $mg\sin\alpha = \frac{mv^2}{r}$ (the component of weight towards O); eliminating v^2: $\sin\alpha = 2(\cos\alpha - \sin\alpha)$, so $\tan\alpha = \frac{2}{3}$. (c) Afterwards the horizontal component $v\sin\alpha$ is conserved; at the horizontal through O the speed satisfies $\frac{1}{2}mW^2 = mgr\cos\alpha$, and $\cos^2\theta = \dfrac{v^2\sin^2\alpha}{W^2} = \dfrac{8}{78} = \dfrac{4}{39}$ – so $\cos\theta = \frac{2}{\sqrt{39}}$.

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Centres of mass 质心 by integration



$$\bar{x} = \frac{\int_0^a x(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx} = \frac{3a}{8}$$



masses 8:1:7; G at $\frac{45a}{112}$ along, $\frac{a}{14}$ off-axis; tilted rest $\tan\theta = \frac{8}{45}$

Left: a hemisphere cross-section with the thin-disc strip at height x and the COM at $3a/8$ from the plane face. Right: the carved solid S (radius a minus radius $a/2$ hemisphere, $OC = a/2$) with its COM G marked along and off the axis.

For a uniform **solid of revolution** 旋转体, $\bar{x} = \dfrac{\int_0^a xy^2\,dx}{\int_0^a y^2\,dx}$ (weight thin discs by $\pi y^2\,dx$). The **hemisphere** 半球 result must be derivable, not just quoted: with $y^2 = a^2 - x^2$, $\int_0^a x(a^2-x^2)dx = \frac{a^4}{4}$ and $\int_0^a (a^2-x^2)dx = \frac{2a^3}{3}$, giving $\bar{x} = \frac{3a}{8}$ from the plane face. **Composite bodies** 组合体: weight the parts by mass (radius-cubed for similar solids) and take moments; removed parts carry negative mass. **Equilibrium** 平衡 of the tilted body: the COM sits vertically above the contact point.

> **Worked example – a carved hemisphere (WME03 June 2025 Q6).** S is a solid hemisphere (radius a) with a radius-$\frac{a}{2}$ hemisphere removed, plane faces coplanar, centres O (big) and C with $OC = \frac{a}{2}$. Masses $8 : 1 : 7$ (radii cubed); along the axis $8 \cdot \frac{3a}{8} - 1 \cdot \frac{3a}{16} = 7\bar{y}$, so the COM of S is $\frac{45a}{112}$ from the OC-line through O. Perpendicular to OC (moments about the in-plane axis through O): $8\cdot 0 - 1\cdot\frac{a}{2} = 7\bar{x}$, so $\bar{x} = \frac{a}{14}$. Resting tilted on its curved surface, $\tan\theta = \dfrac{a/14}{45a/112} = \frac{8}{45}$.

Checklist before you call this unit done

- Convert $a = f(x)$ via $v\,dv/dx$ and close constants from a stated instant; finish the second integration by separation or by eliminating x.
- Use $T = \lambda e/L$ and elastic energy $\lambda e^2/2L$; for midpoint strings treat each half separately; run three-store work-energy with resistance over distance.
- Combine moments with Hooke's law for hinged rods in equilibrium (geometry first: horizontal/vertical offsets of the attachment points).
- Prove SHM by deriving $\ddot{x} = -\omega^2 x$ (equilibrium tensions first), then quote period, $v_{max} = a\omega$ and the sinusoidal time fractions.

- Resolve towards the centre ($r\omega^2$) and vertically for horizontal circles – table reactions, conical pendulums, banked surfaces.
- For vertical circles: energy between levels, $R = 0$ (or $T = 0$) at the leaving point, projectile continuation conserving the horizontal component.
- Derive the hemisphere COM by the disc integral; build composites with signed masses; close tilted equilibrium with the COM-above-contact condition.

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Statistics 1

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Statistics 1: data, probability, regression and distributions

Statistics 1 (S1, paper WST01/01, 1 hour 30 minutes, 75 marks, first assessment June 2019) is the statistics partner of M1 in International AS Mathematics; P1 and P2 are assumed. The formula booklet carries the statistical tables; the formulas below are yours to know. This reference covers data representation and summary, probability, correlation and regression, discrete random variables and the Normal distribution.

Must-know for S1 (not in the booklet):

Family	Formulae
means	$\bar{x} = \dfrac{\sum x}{n}$ or $\dfrac{\sum fx}{\sum f}$
probability	$P(A') = 1 - P(A)$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, $P(A \cap B) = P(A)P(B \mid A)$
coding	$E(aX + b) = aE(X) + b$, $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
Normal	$Z = \dfrac{X - \mu}{\sigma}$ for $X \sim N(\mu, \sigma^2)$

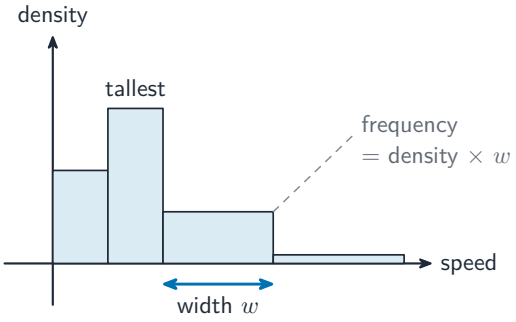
Mathematical models in probability and statistics

A statistical model is a simplified description of a real situation used to represent, analyse and predict. Reasons to use one: it is quicker and cheaper than the real experiment, it can be repeated and refined, and it exposes relationships. Know the standard reasons – "to make predictions" alone is not enough for the mark.

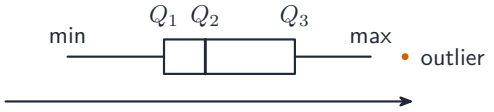
Representing data: histograms, stem and leaf, box plots

In a **histogram** 直方图 the AREA is frequency: bar height is frequency density $\dfrac{\text{frequency}}{\text{class width}}$, so unequal classes get unequal widths, and comparing bars is comparing densities. A **stem and leaf** 茎叶图 keeps the original values (back-to-back compares two groups). A **box plot** 箱线图 shows minimum, lower quartile, median, upper quartile, maximum, with **outliers** 异常值 marked as separate crosses under whatever rule the question states.

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area is frequency: height = f/w for unequal classes



box plot: quartiles and whiskers, outliers as crosses

Left: a histogram of unequal-width classes with frequency density on the vertical axis, one bar annotated frequency = density x width. Right: a box plot with the quartiles labelled and an outlier marked beyond the whisker.

> **Worked example – histogram heights by ratio (WST01 June 2025 Q5a).** Speeds of 225 cars in classes of widths 5, 5, 10, 15, 15 with frequencies 54, 90, 60, 15, 6: the densities are 10.8, 18, 6, 1, 0.4. The tallest bar (25–30, density 18) is drawn 10 cm tall, so the second-tallest (20–25, density 10.8) measures $\frac{10.8}{18} \times 10 = 6$ cm. Beware: the 30–40 bar ALSO has density 6 – only the ratio argument earns the marks, not "60 ÷ 10".

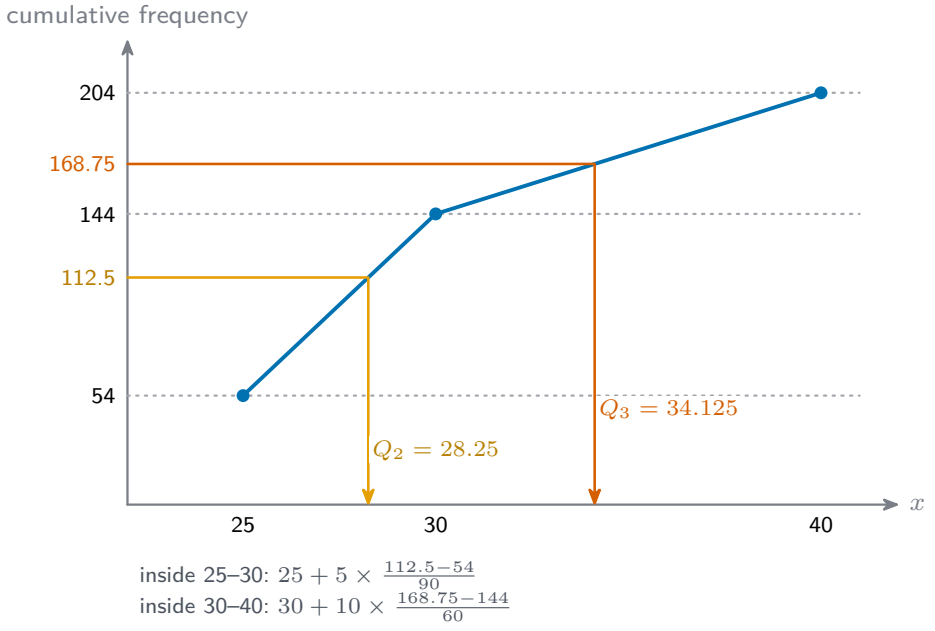
Measures of location and dispersion

For grouped data, estimate the **median** 中位数 or quartiles by linear interpolation inside the containing class: $Q_2 = a + w \times \dfrac{\frac{n}{2} - F}{f}$. The **interquartile range** 四分位距 is $Q_3 - Q_1$; variance is $\sigma^2 = \dfrac{\sum x^2}{n} - \bar{x}^2$ (or $\dfrac{\sum fx^2}{\sum f} - \bar{x}^2$). Combining groups: total $\sum x = \sum n_i \bar{x}_i$, total $\sum x^2$ adds directly, then reapply the formulas. The **standard deviation** 标准差 is the square root of the variance. Coding reverses: if $w = 11.5d$, then $S_{td} = \dfrac{S_{tw}}{11.5}$ and $S_{dd} = \dfrac{S_{ww}}{11.5^2}$.

> **Worked example – combining two classes (WST01 June 2025 Q2).** Class A: $n = 24$, $\bar{x} = 47$, $\sum x^2 = 66876$; class B: $n = 30$, $\bar{x} = b$, $\sum x^2 = 73826$; the 54 combined students average 45. Then $\sum x = 54(45) = 2430$, so $\sum x_B = 2430 - 24(47) = 1302$ and $b = 43.4$. The combined standard deviation: $\sigma = \sqrt{\frac{66876+73826}{54} - 45^2} = \sqrt{580.59\ldots} = 24.1$. If every B mark rises by 2: B's variance is unchanged (shifting moves nothing), the combined mean increases, and the combined SD decreases – B now sits closer to A, so the whole group is less spread.

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Worked example – interpolation quartiles (WST01 June 2025 Q5c–d). With cumulative frequencies 54, 144, 204, 219, 225: the median is the 112.5th value, inside 25–30: $Q_2 = 25 + 5 \times \frac{112.5-54}{90} = 28.25$. Q_3 is the 168.75th, inside 30–40: $Q_3 = 30 + 10 \times \frac{168.75-144}{60} = 34.125$, so with $Q_1 = 25.14$ given, $IQR = 8.985$.



Linear interpolation assumes values spread evenly inside a class: the median 28.25 lies in 25 to 30, the upper quartile 34.125 in 30 to 40

Skewness

Compare distances: $(Q_3 - Q_2)$ versus $(Q_2 - Q_1)$, or mean versus median. Above, $5.875 > 3.11$ (and mean 30.6 > median 28.25), so the speeds are **positively skewed** 右偏 (right-skewed) – which is also why a Normal model would be refused.

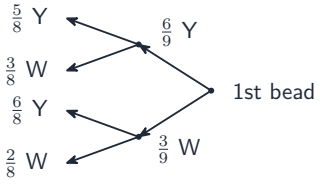
Probability

Work from the **sample space** 样本空间: listing outcomes beats formula-grabbing. For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) = P(A)P(B \mid A)$; **independence** 独立 means $P(B \mid A) = P(B)$, giving the product rule. **Conditional probability** 条件概率 reverses the tree: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ when $P(A) > 0$. Without replacement, reduce the total remaining count by one; reduce a category count only when the item drawn came from that category.

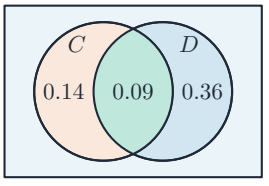
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Dice: equally likely outcomes, the starting point for probability



without replacement: second-layer fractions drop by one top and bottom



$P(C \cup D) = P(C) + P(D) - P(C \cap D)$ fills the regions

Left: a tree diagram for two beads drawn without replacement, first-layer fractions 6/9 and 3/9, second layer 5/8, 3/8 and 6/8, 2/8. Right: a Venn diagram of events C and D with $P(C) = 0.23$, $P(D) = 0.45$, intersection 0.09.

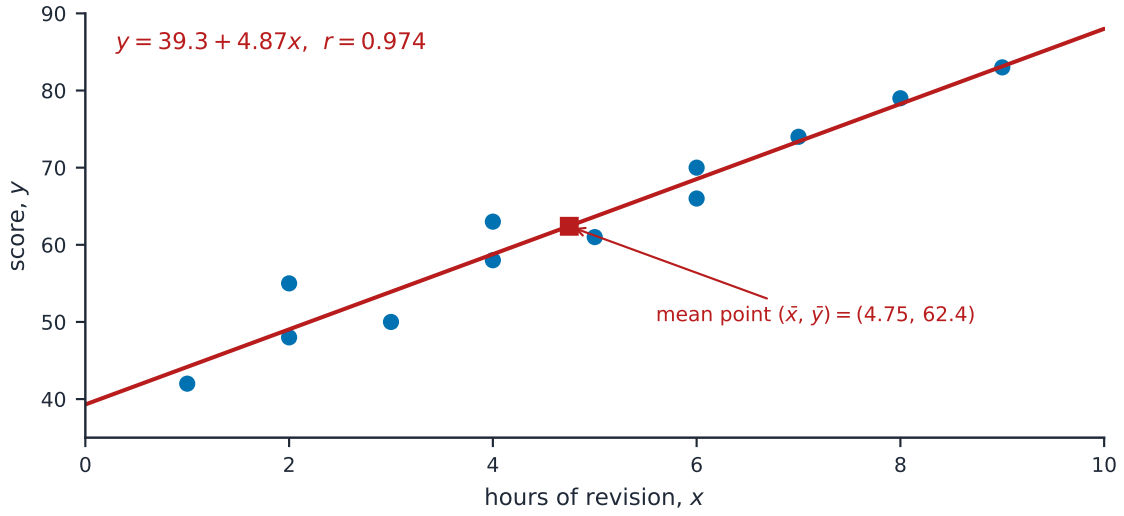
Worked example – Venn to conditional (WST01 June 2025 Q4). $P(C \cup D) = 0.59$, $P(D) = 0.45$, $P(C \mid D) = 0.2$. Then $P(C \cap D) = 0.2 \times 0.45 = 0.09$ and $P(C) = 0.59 + 0.09 - 0.45 = 0.23$ – fill the Venn diagram and both answers fall out.

Worked example – a tree with reversal (WST01 June 2025 Q7). Bag: 3 white, 6 yellow; two drawn without replacement. Tree: $\frac{6}{9}, \frac{3}{9}$, then $\frac{5}{8}, \frac{3}{8}$ after yellow and $\frac{6}{8}, \frac{2}{8}$ after white. Exactly one yellow: $\frac{6}{9} \cdot \frac{3}{8} + \frac{3}{9} \cdot \frac{6}{8} = \frac{1}{2}$. At least one white: $1 - \frac{6}{9} \cdot \frac{5}{8} = \frac{7}{12}$. Given at least one white, both white: $\dfrac{\frac{3}{9} \cdot \frac{2}{8}}{\frac{7}{12}} = \dfrac{1}{12} \cdot \dfrac{12}{7} = \dfrac{1}{7}$.

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Correlation and regression

The **product moment correlation coefficient** 相关系数 $r = \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$ measures LINEAR association ($-1 \le r \le 1$); interpret in context. The least-squares **regression line** 回归线 of y on x is $y = a+bx$ with $b = \dfrac{S_{xy}}{S_{xx}}$ and $a = \bar{y} - b\bar{x}$ – only usable to predict y FROM x, and only within the data range (extrapolation is the flagged danger). Swap x and y and the line is different.



Regression line of y on x through the mean point

> **Worked example – coded regression (WST01 June 2025 Q3).** With $w = 11.5d$, $S_{tt} = 26.43$, $S_{tw} = -91.55$, $\sum w = 339.25$, $\sum t = 9.1$, $\sum w^2 = 10036.45$ over 12 days: $S_{ww} = 10036.45 - \frac{339.25^2}{12} = 445.57$; the pmcc $r = \dfrac{-91.55}{\sqrt{26.43 \times 445.57}} = -0.844$ – warmer days, less gas. The regression of w on t: $b = \frac{-91.55}{26.43} = -3.46$, $a = \frac{339.25}{12} - (-3.46)\frac{9.1}{12} = 30.9$, so $w = -3.46t + 30.9$; divide through by 11.5 for d on t: $d = -0.301t + 2.69$. At $t = 2$: $d = 2.09$ m^3, and each extra degree means about 0.3 m^3 less gas.

Discrete random variables

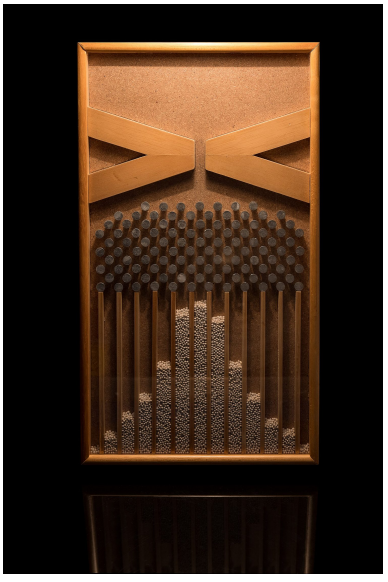
A **discrete random variable** 离散随机变量 takes isolated values with probabilities $p(x) = P(X = x)$ summing to 1; its **probability function** 概率函数 table is the object. **Expectation** 期望 $E(X) = \sum xp(x)$, $E(X^2) = \sum x^2p(x)$, **variance** 方差 $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$. The cumulative function $F(x_0) = P(X \le x_0)$ adds probabilities up to x_0. The discrete UNIFORM distribution on $1, \ldots, n$ has $E(X) = \frac{n+1}{2}$ and $\mathrm{Var}(X) = \frac{n^2-1}{12}$.

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Worked example – the full routine (WST01 June 2025 Q1). X with probabilities 0.3, 0.2, 0.35, 0.15 on 1–4. $F(3.5) = 0.3 + 0.2 + 0.35 = 0.85$. The inequality $3X - 3 < X + 2$ solves to $X < 2.5$, so $P = 0.5$ – solve for X FIRST, then add. $E(X) = 2.35$, $E(X^2) = 6.65$, $\text{Var}(X) = 6.65 - 2.35^2 = 1.1275$; and $\text{Var}(5 - 2X) = 4(1.1275) = 4.51$ by the coding rule.

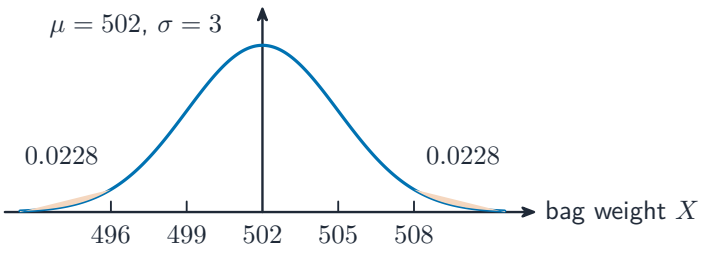
The Normal distribution

For the **Normal distribution** 正态分布 $X \sim N(\mu, \sigma^2)$, standardise: $Z = \dfrac{X - \mu}{\sigma}$ then read the table. Symmetry does half the work: $P(X > \mu + 2\sigma) = P(Z > 2) = 0.0228$ and $P(\mu - 2\sigma < X < \mu + 2\sigma) = 1 - 2(0.0228) = 0.9544$. Reverse problems give z from the table, then $x = \mu + z\sigma$; two conditions on the same k give simultaneous equations.



A Galton board: balls falling through pins pile up into the bell-shaped normal curve

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standardise $Z = \dfrac{X - \mu}{\sigma}$: tails at $z = \pm 2$ hold 0.0228 each

The Normal curve centred at mu = 502 with standard-deviation ticks at 496, 499, 505, 508; the tails beyond 496 and 508 shaded (about 2.28 percent each), and the standardising arrow Z = (X minus mu)/sigma.

Worked example – bags of flour (WST01 June 2025 Q6). Small bags $X \sim N(502, 3^2)$: $P(X > 508) = P(Z > 2) = 0.0228$. Of 4 bags, exactly 2 over 508: $6(0.0228)^2(0.9772)^2 = 0.00298$. Large bags $Y \sim N(1024, \sigma^2)$ with 85th percentile 1038.51: $z_{0.85} = 1.0364$, so $\sigma = \frac{14.51}{1.0364} = 14$. If $P(X > k) = P(Y < 2k)$, then $\frac{k-502}{3} = \frac{1024-2k}{14}$, so $14k - 7028 = 3072 - 6k$, $k = 505$, and $p = P(Z > 1) = 0.159$.

Checklist before you call this unit done

- Give several reasons why statistical models are used; keep "cheap and quick" plus one more.
- Compute frequency densities for unequal classes; compare histogram bars by ratio; interpolate quartiles inside the containing class; mark outliers under the stated rule.
- Combine data sets through $\sum x$ and $\sum x^2$; reverse coding on S statistics; predict the effect of shifting or scaling on mean, variance and SD.
- Solve probability questions from Venn diagrams and trees, with and without replacement; reverse conditionals with $P(A \cap B)/P(A)$.
- Compute r from S statistics and interpret it; fit $y = a + bx$; predict only within range; pass coded data back to the original variable.
- Run the discrete DRV routine: probabilities sum to 1, solve inequalities for X first, $E(X)$, $E(X^2)$, variance, linear coding; quote the uniform results.
- Standardise both directions for Normal questions; use symmetry; finish two-condition problems with simultaneous equations.

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Statistics 2

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Statistics 2: distributions, continuous variables and hypothesis tests

Statistics 2 (S2, paper WST02/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) follows S1 in the applied route: P1–P2 and S1 are assumed. The booklet carries the binomial, Poisson and Normal tables. This reference covers the binomial and Poisson distributions with their approximations, continuous random variables, the uniform distribution, sampling and the full hypothesis-testing routine.

Must-know for S2 (not in the booklet):

Family	Formulae
binomial	$X \sim B(n,p)$: $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$
Poisson	$X \sim \mathrm{Po}(\lambda)$: $E(X) = \mathrm{Var}(X) = \lambda$; rates scale with time
continuous	$P(a<X<b) = \int_a^b f(x)\,dx$, $f(x) = F'(x)$, $E(X) = \int x f(x)\,dx$, $E(X^2) = \int x^2 f(x)\,dx$
uniform	on $[a,b]$: mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$, density $\frac{1}{b-a}$

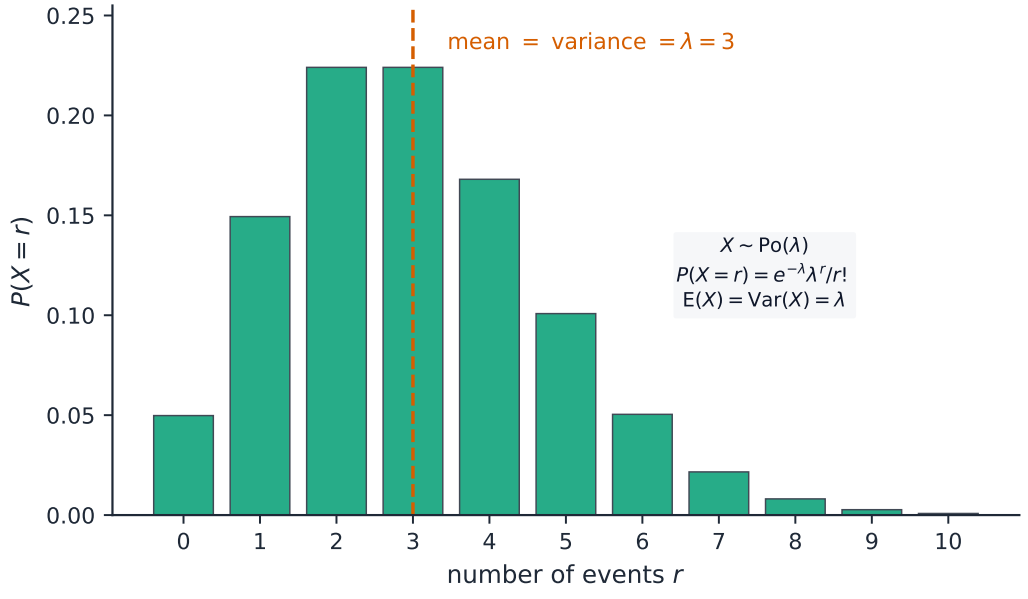
The binomial and Poisson distributions

Binomial distribution 二项分布: n independent trials, constant probability p – the two assumptions to state in context. **Poisson distribution** 泊松分布: events occur singly, independently, at a constant average rate – the rate scales ($\mathrm{Po}(\lambda)$ per minute gives $\mathrm{Po}(5\lambda)$ in 5 minutes). Moments come as formulas: np and $np(1-p)$; λ and λ. Given the mean and variance of a binomial, solve for n and p simultaneously. The **Poisson approximation** to the binomial needs n large and p small (use $\lambda = np$).

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People arriving at random at a steady average rate: the number per minute follows a Poisson distribution

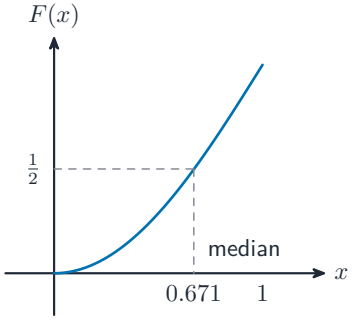


The Poisson distribution Po(3)*: for a Poisson variable the mean and variance both equal* λ*.*

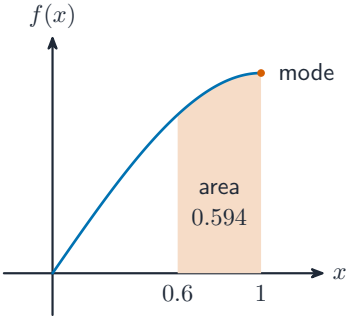
Worked example – moments then approximations (WST02 June 2025 Q3). Blue sweets per small bag have mean 3 and variance 2.55: $np = 3$ and $np(1-p) = 2.55$ give $1-p = 0.85$, so $p = 0.15$ and $n = 20$. A medium bag of 40 with 20% green: $X \sim B(40, 0.2)$, $E(X) = 8$, and $P(X > 8) = 1 - P(X \leq 8) = 0.407$. A large bag of 80 with 10% yellow: with $n = 80$ large and $p = 0.1$ small, $X \approx \text{Po}(8)$ and $P(X \geq 12) = 1 - P(X \leq 11) = 0.112$.

Continuous random variables

A **continuous random variable** has a **probability density function** 概率密度函数 f with total area 1 and $P(X = x) = 0$ at every point; the **cumulative distribution function** 分布函数 $F(x) = P(X \leq x)$ satisfies $f = F'$ and $F(a) = \int_{-\infty}^{a} f$. Piecewise densities: normalise $\int f = 1$ (which pins the unknown constant), then read off the **median** 中位数 ($F = \frac{1}{2}$), quartiles, and the **mode** 众数 from the peak of f (set $f' = 0$, or inspect corners). $E(X) = \int x f\,dx$, $E(X^2) = \int x^2 f\,dx$, $\text{Var} = E(X^2) - [E(X)]^2$; skewness compares mean, median and mode.



median solves $F(m) = \frac{1}{2}$;
mode differentiates $f = F'$

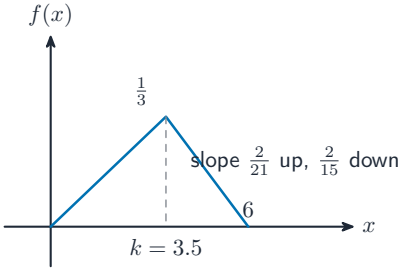


$P(X > 0.6) = 1 - F(0.6)$;
peak of f at the boundary

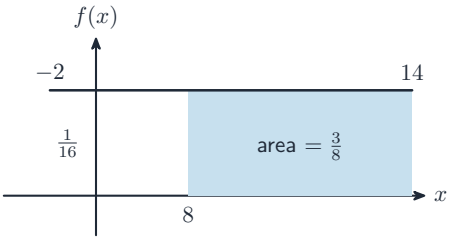
Left: the CDF F(x) = (6x^2 - x^4)/5 on [0, 1] rising from 0 to 1 with the median marked at F = 1/2 (x = 0.671). Right: the density f(x) = (12x - 4x^3)/5 with the mode at x = 1 and the area beyond x = 0.6 shaded (0.594).

Worked example – a CDF worked through (WST02 June 2025 Q1). $F(x) = k(6x^2 - x^4)$ on $[0, 1]$: $F(1) = 1$ gives $5k = 1$. $P(X > 0.6) = 1 - \frac{6(0.36) - 0.1296}{5} = 0.594$. The median: $\frac{6m^2 - m^4}{5} = \frac{1}{2}$, i.e. $2m^4 - 12m^2 + 5 = 0$, so $m = 0.671$. The mode: $f(x) = \frac{12x - 4x^3}{5}$ and $f'(x) = \frac{12 - 12x^2}{5} = 0$ gives $x = 1$ (reject -1, outside the support). Skewness: mean $0.64 <$ median $0.671 <$ mode 1 – negative skew.

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normalising pins k: $(2k-7)^2 = 0$, continuous at the peak



uniform: probabilities are lengths over $b - a = 16$

Left: a triangular pdf rising along 2x/21 from 0 to k = 3.5 then falling along 2(6-x)/15 to x = 6, peak value 1/3 marked. Right: the uniform density on [-2, 14] at height 1/16 with the interval [8, 14] shaded (area 3/8).

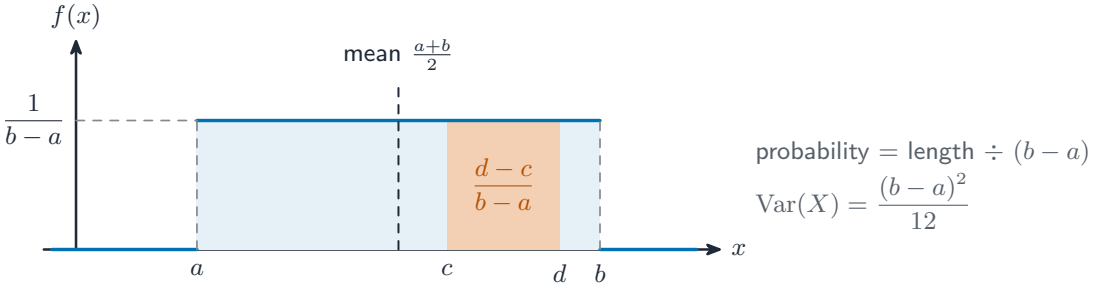
Worked example – a piecewise-linear pdf, no calculator (WST02 June 2025 Q5). With $f(x) = \frac{2x}{21}$ on $[0, k]$ and $f(x) = \frac{2(6-x)}{15}$ on $(k, 6]$: normalising,

$$\frac{k^2}{21} + \frac{2}{15}\left(18 - 6k + \frac{k^2}{2}\right) = 1 \implies 4k^2 - 28k + 49 = 0 \implies (2k-7)^2 = 0,$$

so $k = 3.5$ (the pdf is continuous there, both pieces equalling $\frac{1}{3}$). Then $E(X) = \frac{2}{21}\cdot\frac{k^3}{3} + \frac{2}{15}\int_{3.5}^{6}(6x - x^2)dx = \frac{19}{6}$, and with $E(X^2) = \frac{277}{24}$ given, $\mathrm{Var}(X) = \frac{277}{24} - \frac{361}{36} = \frac{109}{72}$.

The continuous uniform distribution

The **continuous uniform distribution** 均匀分布 on $[a, b]$ Uniform on $[a, b]$: density $\frac{1}{b-a}$, straight CDF, mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$. Two facts pin a and b (an interval probability gives $b - a$; the mean gives $a + b$). Coding scales the mean; probabilities are lengths over $b - a$.



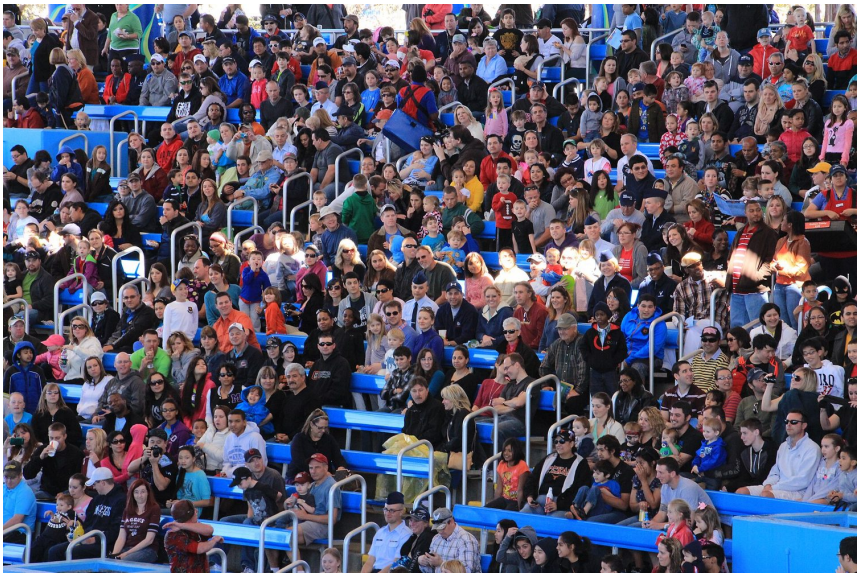
Uniform on [a, b]: probabilities are lengths divided by b − a

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Worked example – uniform on a wide interval (WST02 June 2025 Q6). Assume $a \leq 2$ and $b \geq 5$. Then $P(2 < X < 5) = \frac{3}{16}$ gives $b - a = 16$; $E(X) = 6$ gives $a + b = 12$: so $a = -2$, $b = 14$ Without that support assumption, $a = 4.4$, $b = 7.6$ also satisfies the two given facts; the following calculations use the stated support assumption. $E(cX + 1) = 3$ gives $c = \frac{1}{3}$; Var $= \frac{256}{12} = \frac{64}{3}$; $E(X^2) = \frac{64}{3} + 36 = \frac{172}{3}$; and $P(\frac{3}{2}X - b > a) = P(X > \frac{2(a+b)}{3}) = P(X > 8) = \frac{14-8}{16} = \frac{3}{8}$.

Populations, samples and statistics

A **census** 普查 aims to measure everybody (complete but costly, slow, possibly destructive). To **sample** 抽样 is to measure a selected part of the population; the selected units are also called a sample (cheaper and faster, but with sampling error). The **sampling frame** 抽样框 lists the sampling units. A **statistic** 统计量 is a function of the sample values only (no unknown parameters) – $\bar{X}$ is, $\bar{X} - \mu$ is not; its distribution over repeated samples is the **sampling distribution** 抽样分布.



Statistics studies a sample to learn about a whole population.

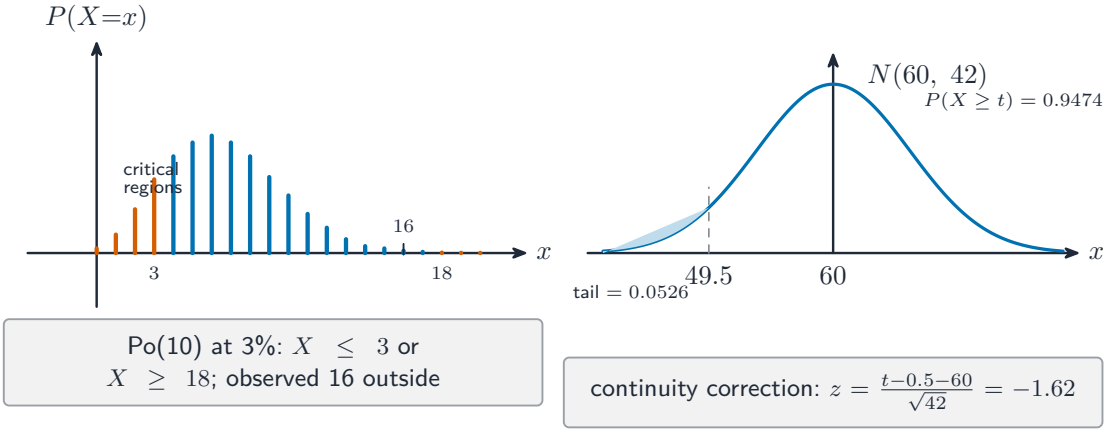
Worked example – a sampling distribution (WST02 June 2025 Q4). Three beads from a 2:5 black:white bag, scoring 4 per black and 1 per white: with $P(\text{black}) = \frac{2}{7}$, the score X takes 3, 6, 9, 12 with probabilities $\frac{125}{343}, \frac{150}{343}, \frac{60}{343}, \frac{8}{343}$ (binomial weights $\binom{3}{0}(\frac{5}{7})^3$ etc.). If Y counts 9-point sets among n independent plays, $Y \sim B(n, \frac{60}{343})$ and $1 - (\frac{283}{343})^n > 0.95$ needs $n > 15.6$, so $n = 16$.

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Hypothesis tests

A **hypothesis test** 假设检验 compares data against a null model.

State the **null hypothesis** 原假设 H_0 (no change, the quoted parameter) and H_1 (one- or two-tailed as asked). Find the **critical region** 临界域 at the stated nominal level by choosing tails whose total null probability does not exceed α; report the actual **significance level** 显著性水平 as the total tail probability. One-tailed tests put the whole level in one tail; two-tailed split it. Conclude IN CONTEXT: "insufficient evidence that … has changed". Normal approximations to the binomial: mean np, variance $np(1-p)$, with the **continuity correction** 连续性修正 (at least t means $t-0.5$).



Left: the Po(10) distribution as vertical bars with the critical regions X <= 3 and X >= 18 in a contrasting colour and the observed 16 marked between them. Right: the N(60, 42) curve with the area below 49.5 shaded and the continuity-corrected z = -1.62 marked.

Worked example – a Poisson rate test (WST02 June 2025 Q2). Customers average 20/hour; after alterations one 30-minute interval is watched: $X \sim \text{Po}(10)$, $H_0\colon \lambda = 10$, $H_1\colon \lambda \neq 10$. At 3%: $P(X \leq 3) = 0.0103$ but $P(X \leq 4) = 0.0293$ (too big); $P(X \geq 17) = 0.0270$ but $P(X \geq 18) = 0.0143$ – critical region $X \leq 3$ or $X \geq 18$, actual level $0.0103 + 0.0143 = 2.46\%$. The observed 16 lies outside → do not reject H_0.

Worked example – binomial test with a Normal middle (WST02 June 2025 Q7). 30% of customers take insurance; a sample of 40 gives $X \sim B(40, 0.3)$. Largest r with $P(X < r) < 0.05$ (fewer than r): $P(X \leq 6) = 0.0238 < 0.05$ but $P(X \leq 7) = 0.0553 > 0.05$, so $r = 7$. For 200 customers, $B(200, 0.3) \approx N(60, 42)$: $P(X \geq t) = 0.9474$ means $\frac{t-0.5-60}{\sqrt{42}} = -1.62$, so $t = 50$. After a special offer, 11 of 25 buy: $H_0\colon p = 0.3$, $H_1\colon p > 0.3$; $P(X \geq 11) = 0.0978 < 0.10$, so reject H_0 – evidence the proportion has increased.

Checklist before you call this unit done

- State binomial/Poisson assumptions in context; solve mean/variance pairs for n, p; scale Poisson rates with time.
- Normalise piecewise densities (constants, perfect-square k); differentiate F for f; find median/quartiles from F, mode from f' or corners; run E(X), E(X^2), Var; read skewness from mean–median–mode.
- Quote and derive the uniform facts; pin a and b from an interval plus the mean; keep probabilities as lengths.
- Distinguish census vs sample with pros and cons; recognise statistics; build small sampling distributions by enumeration; wrap counts in B(n, p) for follow-ups.
- Run one- and two-tailed tests: hypotheses in symbols and words, critical region by accumulation, actual significance level, in-context conclusion; Normal approximation with continuity correction when n is large.

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Statistics 3

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Statistics 3: samples, estimation, chi-squared and correlation

Statistics 3 (S3, paper WST03/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the final statistics unit: S1–S2 are assumed. The booklet carries the Normal and chi-squared tables plus the Spearman critical values. This reference covers combinations of Normal variables, sampling methods, estimators and the sample mean's distribution, confidence intervals and **hypothesis tests** 假设检验 for means (one and two samples), goodness of fit and contingency tables, and rank correlation with its test.

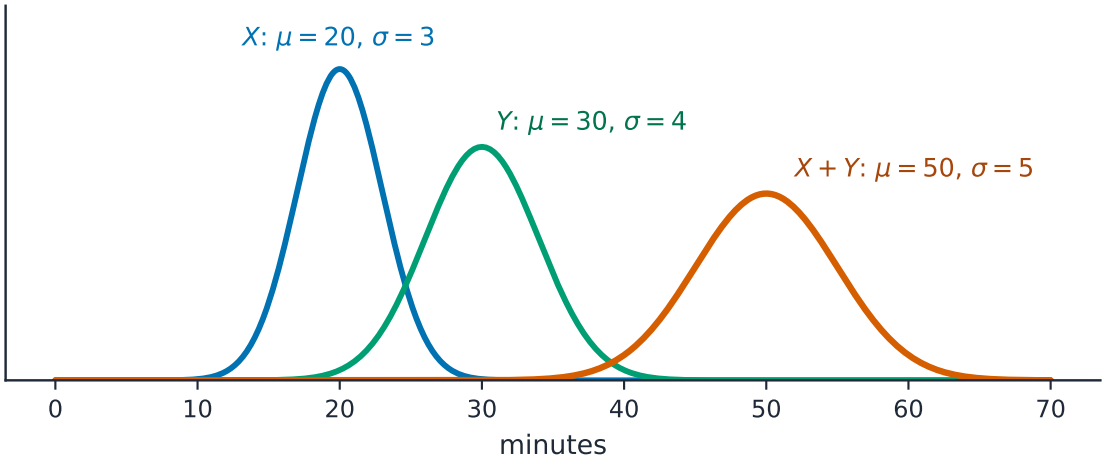
Must-know for S3 (not in the booklet):

Family	Formulae
combinations	$X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$ independent $\Longrightarrow aX \pm bY \sim N(a\mu_x \pm b\mu_y,\ a^2\sigma_x^2 + b^2\sigma_y^2)$
sample mean	for independent observations from a Normal population, $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$; $\text{SE} = \dfrac{\sigma}{\sqrt{n}}$
unbiased variance	$s^2 = \dfrac{\sum(x_i - \bar{x})^2}{n-1} = \dfrac{\sum x_i^2 - n\bar{x}^2}{n-1}$
test statistics	$z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$; two samples $z = \dfrac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sqrt{\sigma_x^2/n_x + \sigma_y^2/n_y}}$
chi-squared	$X^2 = \sum \dfrac{(O-E)^2}{E}$, $E = \dfrac{\text{row total} \times \text{col total}}{\text{grand total}}$
Spearman	$r_s = 1 - \dfrac{6\sum d^2}{n(n^2-1)}$

Combinations of random variables

Independent Normal variables combine to a Normal: add means, add **squared** coefficients times variances. For $3S$ where $S \sim N(18, 0.03^2)$: $3S \sim N(54, 9 \times 0.0009)$ – the mean triples but the variance multiplies by nine. Differences work the same way: $C_1 - C_2 \sim N(0, 2\sigma^2)$, and $P(|C_1 - C_2| > c)$ is twice the one-tail probability by symmetry.

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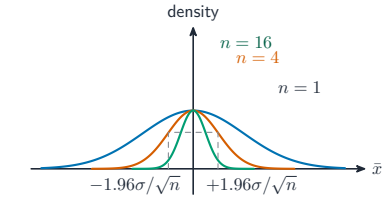


$$\mu_{X\pm Y}=\mu_X\pm\mu_Y \quad \sigma^2_{X\pm Y}=\sigma^2_X+\sigma^2_Y,\ \text{so}\ \sigma_{X+Y}=\sqrt{3^2+4^2}=5$$

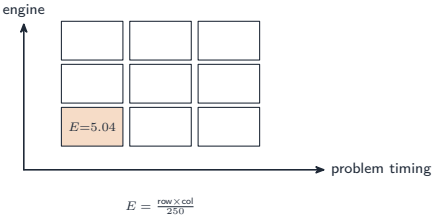
For independent random variables means add and variances add

Worked example (WST03 June 2025 Q8). $S\sim N(18,0.03^2)$, $C\sim N(25,0.03^2)$, $P\sim N(15,0.2^2)$ independently (kg). (a) $S_1+S_2+S_3\sim N(54,0.0027)$ (sd 0.05196): $P(>54.1)=1-\Phi(1.924)=0.0274$. (b) $C_1-C_2\sim N(0,0.0018)$: $P(|C_1-C_2|>0.02)=2(1-\Phi(0.4714))=2\times0.3192=0.638$. (c) With $T=\sum_1^{28}S_i+\sum_1^5 C_i+P_1$, the event $T<30P_1+190$ rearranges to $\sum S_i+\sum C_i-29P_1<190$ – a Normal with mean $504+125-435=194$ and variance $28(0.0009)+5(0.0009)+29^2(0.04)=33.6697$, so $P=\Phi\left(\frac{190-194}{5.803}\right)=1-0.7549=0.245$.

Sampling methods



SE $=\sigma/\sqrt{n}$: the interval shrinks like $1/\sqrt{n}$



$E=\frac{\text{row}\times\text{col}}{250}$

$X^2=\sum\frac{(O-E)^2}{E}=12.3$ on $(3-1)(3-1)=4$ df

Left: the sampling distribution of the sample mean for n = 1, 4, 16 - the same mean with standard error shrinking like 1 over root n - and the 95 percent interval marked. Right: a 3-by-3 contingency-table sketch with the diesel E = 5.04 cell highlighted and the chi-squared formula.

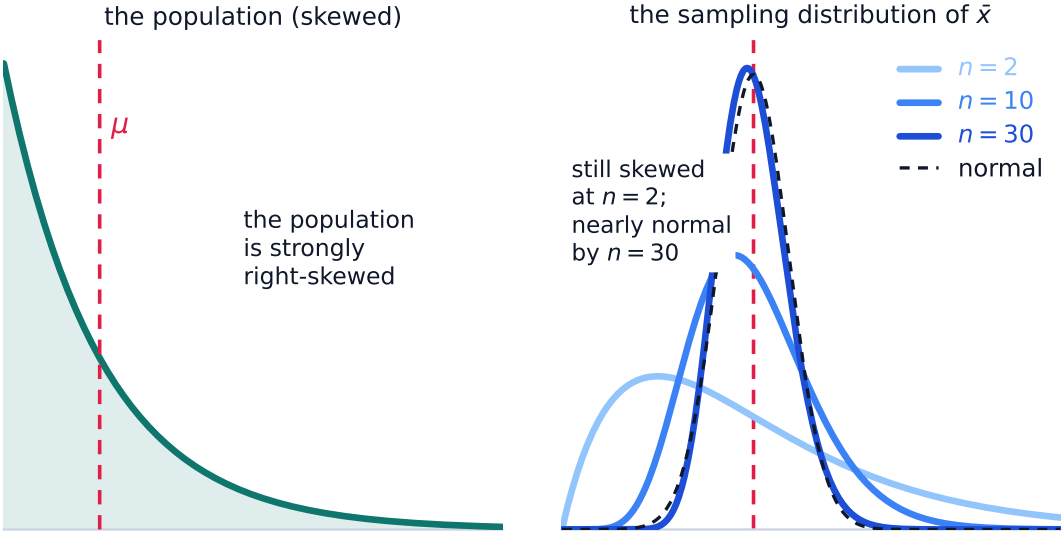
The **simple random sample** 简单随机抽样 gives every member (and every sample

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of size n) an equal chance; use random numbers on a sampling frame. **Systematic sampling** 系统抽样 takes every xth member from a random start – for 20 from 360, $x = 18$; it avoids choosing individuals by personal preference, but periodic patterns in the list can distort the sample, but members in the same stretch can never appear together (spacings are locked). **Stratified sampling** 分层抽样 splits by strata and samples each proportionally – often useful when groups differ from each other and values within each group are relatively similar. **Quota sampling** 配额抽样 fills interviewer-set quotas without a frame – cheap but vulnerable to interviewer bias. Know one advantage and one disadvantage of each.

Estimators, bias and the sample mean

A **statistic** 统计量 contains no unknown population parameters. An estimator is **unbiased** 无偏 if its expectation equals the parameter; its **bias** 偏差 is the gap: $\bar{X}$ for μ, $s^2 = \dfrac{\sum x_i^2 - n\bar{x}^2}{n-1}$ for σ^2 (the $n-1$ is the correction). Bias is $\mathrm{E}(\hat{\theta}) - \theta$ in sign and size. For independent observations from a Normal population, the sample mean satisfies $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$: bigger samples shrink the **standard error** 标准误 $\sigma/\sqrt{n}$.



CLT: for large n, $\bar{x} \approx \text{Normal}(\mu,\ \sigma/\sqrt{n})$ even when the population is not normal

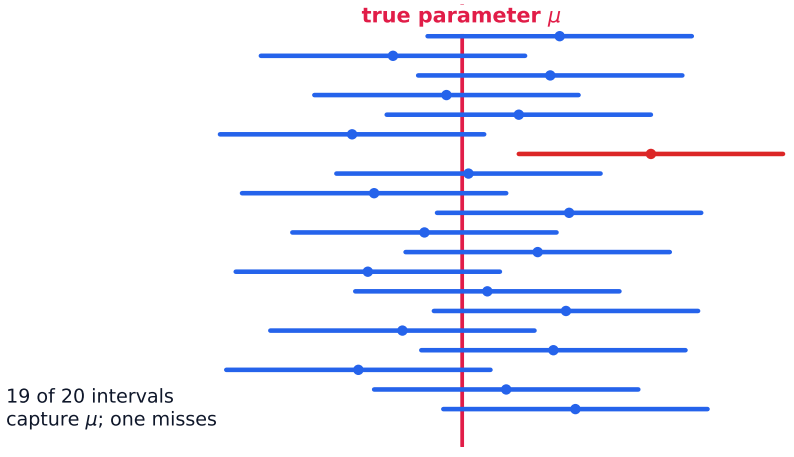
The population on the left is strongly skewed, yet every sampling distribution of $\bar{x}$ is centered at μ. A larger n shrinks the standard error $\sigma/\sqrt{n}$, so the curve gets taller and narrower – and it also straightens: still clearly skewed at $n = 2$, almost exactly normal (dashed) by $n = 30$.

Worked example (WST03 June 2025 Q6). $R = \frac{1}{2}X_1 + \frac{3}{4}X_{20}$: $\mathrm{E}(R) = (\frac{1}{2}+\frac{3}{4})\mu = \frac{5}{4}\mu \neq \mu$, so R is biased with bias $\frac{\mu}{4}$. Given $S = \frac{aX_1 - bX_2}{2}$ and $T = \frac{a\sum_1^{14} X_i + b\sum_{15}^{20} X_i}{20}$ both unbiased: the expectations force $a - b = 2$ and $14a + 6b = 20$, so $a = 1.6$, $b = -0.4$.

Worked example – recovering parameters (WST03 June 2025 Q7). Given $P(\bar{X}_n < 45.2) = 0.3446$ and $P(\bar{X}_n > 75.2) = 0.0179$: the z-values are -0.4 and 2.1, so $\mu - 0.4\frac{\sigma}{\sqrt{n}} = 45.2$ and $\mu + 2.1\frac{\sigma}{\sqrt{n}} = 75.2$. Subtracting: $2.5\frac{\sigma}{\sqrt{n}} = 30$, giving $\sigma = 12\sqrt{n}$ (SE 12) and $\mu = 45.2 + 4.8 = 50$; then $P(\bar{X}_n > 59) = 1 - \Phi(0.75) = 0.2266$.

Confidence intervals and tests for a mean

A **confidence interval** 置信区间 for μ (variance known) is $\bar{x} \pm z\frac{\sigma}{\sqrt{n}}$ with $z = 1.96$ (95%), 1.6449 (one-tailed 95% critical), 2.5758 (99%). Interpretation: the procedure captures μ in 95% of samples – not "the probability that μ lies inside is 0.95". The test statistic $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$; when the variance is unknown and n is large, substitute s (the **Central Limit Theorem** 中心极限定理 also licenses non-Normal populations under independent sampling from the same population with finite, nonzero variance when the sample is sufficiently large). Conclusions state the evidence in context.



CI form: $\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$ · level = long-run capture rate of the method

"95% confident" describes the method, not one interval: over many samples about 95% of the intervals contain μ and about 5% miss it.

Worked example – two samples with a gap (WST03 June 2025 Q4). Grey squirrels ($n = 80$, $\bar{x} = 500$, $s_x^2 = 1156$) and red ($n = 80$; $\sum y = 23200$, $\sum y^2 = 6741351$, giving $\bar{y} = 290$ and $s_y^2 = \frac{6741351 - 80(290)^2}{79} = 169$). Test H_0: $\mu_x - \mu_y = 200$ against > 200 at 5%:

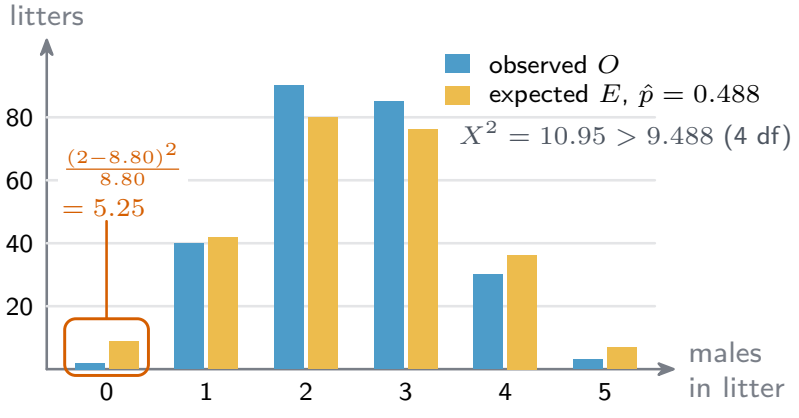
$$z = \frac{(500 - 290) - 200}{\sqrt{1156/80 + 169/80}} = \frac{10}{4.0698} = 2.46 > 1.6449,$$

so reject H_0 – the grey mean is more than 200 g above the red. The independent samples are large, so sample variances estimate the unknown population variances in this approximate test; this does not assert exact equality $\sigma^2 = s^2$.

Goodness of fit and contingency tables

Goodness of fit 拟合优度检验 compares observed counts with a model's expected ones: $X^2 = \sum \frac{(O-E)^2}{E}$. Degrees of freedom = (cells − 1) − (parameters estimated from the data): a binomial with p̂ estimated loses one more. Combine tail cells whose **expected frequency** 期望频数 is below 5 before anything else. **Contingency tables** 列联表 test association between two classifications with $E = \frac{\text{row} \times \text{column}}{\text{total}}$ and $\text{df} = (r-1)(c-1)$.

Worked example – binomial fit (WST03 June 2025 Q5). Males in 250 litters of 5 ($O = 2, 40, 90, 85, 30, 3$): $\hat{p} = \frac{610}{1250} = 0.488$, giving expected frequencies $8.80, 41.92, 79.91, 76.16, 36.30, 6.92$. The r-class contributes $\frac{(2-8.796)^2}{8.796} = 5.25$, so with the given $\sum = 5.70$ for the rest, $X^2 = 10.95$ on **degrees of freedom** 自由度 $6 - 1 - 1 = 4$; the 5% critical value 9.488 is exceeded – reject: the binomial is not a suitable model.



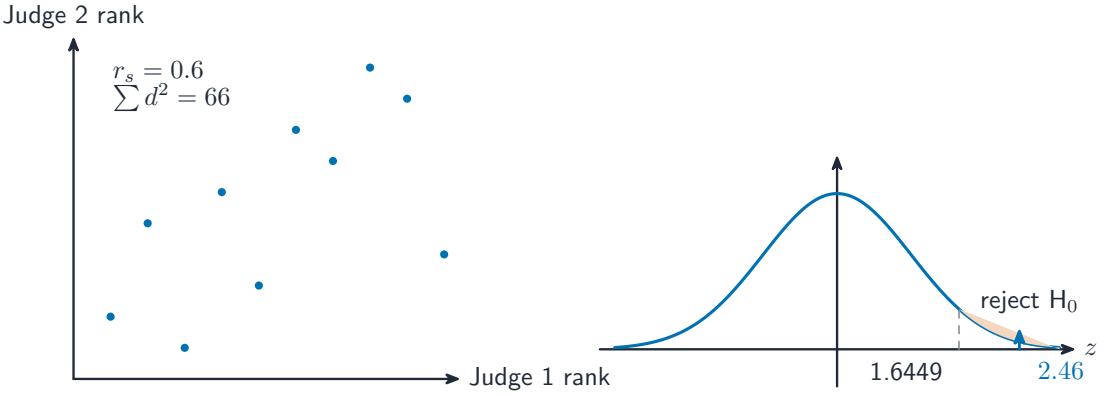
Observed against binomial expected counts: the 0-male class alone contributes 5.25, and X squared = 10.95 exceeds 9.488 on 4 degrees of freedom

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Worked example – association (WST03 June 2025 Q3). Engine type × first-problem timing (250 cars, 3×3): the diesel/month 1–6 cell has $E = \frac{35\times36}{250} = 5.04$ and contributes $\frac{(10-5.04)^2}{5.04} = 4.88$; with 7.444 for the other eight cells, $X^2 = 12.325 > 9.488$ on 4 df – evidence of an association between engine type and when problems start.

Rank correlation and its test

Spearman's rank correlation coefficient measures monotonic agreement between two rankings; test $H_0\colon \rho = 0$ against the table and conclude in words about the rankings.



positive but short of the 1% critical value 0.7455

one-tailed 5%: $z = 2.46 >$ 1.6449 — the gap exceeds 200 g

Left: a scatter of two judges' ranks of 10 performances with r_s = 0.6 noted. Right: the standard normal curve with the one-tailed 5 percent critical value 1.6449 and the observed z = 2.46 marked in the rejection region.

Spearman's rank correlation coefficient 斯皮尔曼等级相关系数 $r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$ measures monotonic agreement between two rankings (no ties examined). Test $H_0\colon \rho = 0$ against one- or two-tailed alternatives using the table of critical values; conclude in context ("insufficient evidence of positive agreement between the judges").

Worked example (WST03 June 2025 Q2). Two judges rank 10 performances: the rank differences give $\sum d^2 = 66$, so $r_s = 1 - \frac{6(66)}{990} = 0.6$. At the 1% one-tailed level the **critical value** 临界值 is 0.7455: $0.6 < 0.7455$, so do not reject H_0 – insufficient evidence of positive correlation between the judges' rankings.

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Checklist before you call this unit done

- Combine independent Normals (squared coefficients on variances); handle differences and |differences| by symmetry.
- Name and justify the four sampling methods; compute systematic spacing and its impossibility properties.
- Check statistics for parameters, compute expectations for bias, solve paired unbiased-estimator equations.
- Use SE $\sigma/\sqrt{n}$; recover (μ, σ) from two sample-mean probabilities; compute probabilities for $\bar{X}$.
- Build and interpret z-based confidence intervals; run one- and two-sample tests with variance substitution and CLT stated.
- Compute expected frequencies (model or contingency), combine low cells, subtract estimated parameters from the df, compare with the χ^2 table.
- Compute Spearman's r_s from ranks and test against the table with hypotheses stated in symbols and the conclusion in words.

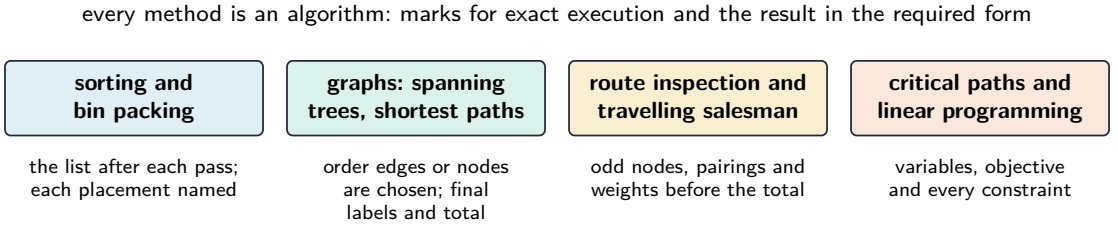
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Decision Mathematics 1

Pearson Edexcel · International A-Level · Further Mathematics

Decision Mathematics 1: algorithms, networks and linear programming

Decision Mathematics 1 (D1, paper WDM01/01, 1 hour 30 minutes, 75 marks, first assessment June 2019) is the discrete option of the IAL: P1–P2 are assumed, and every method is an ALGORITHM – a finite list of exact instructions. No formula booklet entries beyond the basics: the marks are for executing procedures precisely and stating results in the required form. This reference covers sorting and packing algorithms, minimum spanning trees, shortest paths, the Chinese postman and travelling salesman problems, critical path analysis and linear programming.



Every D1 method is an algorithm: the marks are for exact steps in the required form

House rules for algorithm answers:

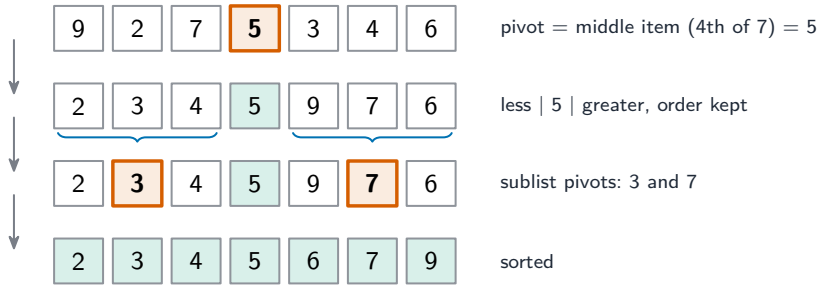
Task	Required form
sorting	list the list after EACH pass (or pivot choices for quick sort)
bin packing	name the algorithm (first-fit, first-fit decreasing, full-bin) and show each placement
graph algorithms	state the order edges/nodes are chosen; show final values (labels, total weight)
route inspection	list the odd nodes, the pairings and their weights, before totalling
linear programming	define variables, objective and EVERY constraint with non-negativity

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Sorting and packing algorithms

The **bubble sort** 冒泡排序 compares adjacent items, swapping out-of-order pairs, one pass at a time; a pass with no swaps ends it. The **quick sort** 快速排序 picks the MIDDLE item of the list as pivot, partitions the remaining items into "less than pivot", "equal to pivot" and "greater than pivot" groups (keeping every copy); it recursively sorts the smaller and larger groups – always quote the pivot and the list after each partition. The spec pins the middle-item choice to the method defined in its glossary, so exercise lists here keep an odd number of items and the middle item is the one at position $\frac{n+1}{2}$. **Bin packing** 装箱问题: first-fit places each item in the first bin with room; first-fit decreasing sorts items largest-first then applies first-fit; full-bin packing looks for combinations filling a bin exactly.

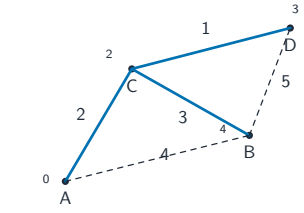
> **Worked example – quick sort (original).** Sort 9, 2, 7, 5, 3, 4, 6. The middle item (position 4 of 7) is 5: less = 2, 3, 4; greater = 9, 7, 6, giving 2, 3, 4, **5**, 9, 7, 6. Sublist 2, 3, 4 (pivot 3): 2, **3**, 4 – sorted. Sublist 9, 7, 6 (pivot 7): 6, **7**, 9 – sorted. Final list: 2, 3, 4, 5, 6, 7, 9.



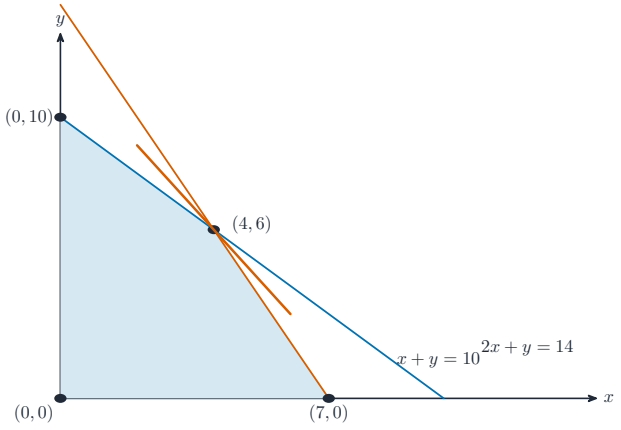
Quick sort with the middle-item pivot: quote the pivot and the list after every partition

> **Worked example – bin packing (original).** Bins of capacity 10; items 7, 5, 4, 3, 6, 2 (total 27, so at least 3 bins). First-fit: 7→b1, 5→b2 (7+5 > 10), 4→b2 (9), 3→b1 (10), 6→b3, 2→b3 (8): bins (7,3), (5,4), (6,2). First-fit decreasing: 7, 6, 5, 4, 3, 2: 7→b1, 6→b2, 5→b3 (6+5 > 10), 4→b2 (10), 3→b1 (10), 2→b3 (7): bins (7,3), (6,4), (5,2) – also 3 bins.

Minimum spanning trees



MST = AC + CD + CB = 6; Dijkstra: A 0, C 2, B 4, D 3 via A–C–D



$P = 3x + 2y$: vertices give 0, 21, 24, 20 — max at $(4, 6)$

Left: a weighted network on A, B, C, D with the MST edges AC 2, CD 1, CB 3 highlighted and Dijkstra labels 0, 2, 3, 8 at the vertices. Right: the feasible region of a linear program with vertices (0,0), (7,0), (4,6), (0,10) and the objective line touching the optimum at (4,6).

A **minimum spanning tree (MST)** 最小生成树 connects every vertex with the least total weight. **Kruskal's algorithm** 克鲁斯卡尔算法: sort edges by weight, add the smallest that does NOT form a cycle (reject and move on when it would). **Prim's algorithm** 普里姆算法: grow a tree from any start vertex, repeatedly adding the cheapest edge from the tree to a new vertex. The matrix form of Prim: cross out the start row, ring the cheapest remaining entry in the column of the chosen vertex, and continue. Kruskal is edge-driven (good on a drawn network); Prim is vertex-driven (good on a matrix). Both give the same total; edges may differ only when weights tie.

> **Worked example (original).** Edges AB 4, AC 2, BC 3, BD 5, CD 1: Kruskal takes CD 1, AC 2, BC 3 (B joins – no cycle), then rejects AB 4 and BD 5 (cycles): MST total 6. Prim from A: AC 2, CD 1 (cheapest edge from {A,C} to a new vertex), CB 3: the same tree, and the total 6 ✓. Always list the edges in the order chosen and give the total.

Shortest paths

For non-negative edge weights, **Dijkstra's algorithm** 迪杰斯特拉算法 labels vertices with the smallest total distance found so far: start vertex label 0; repeatedly pick the unlabelled vertex of smallest temporary label, make it permanent, and update its neighbours (new = permanent + weight, only if smaller). Record the order of permanence and the final labels; the route is recovered by tracing back through vertices whose label equals the final label minus the connecting weight.



A metro map is a network: Dijkstra's algorithm finds the shortest route between two stations

> **Worked example (original).** Edges AB 4, AC 2, BC 1, BD 5, CD 8. Labels: A = 0 permanent; C = 2 permanent; B = min(4, 2+1) = 3 permanent; D = min(3+5, 2+8) = 8 permanent. The shortest route from A to D has length 8: A–C–B–D (trace back: 8 = 3 + 5 at B, 3 = 2 + 1 at C).

The Chinese postman problem

The **route inspection (Chinese postman) problem** 中国邮递员问题 walks every EDGE at least once and returns to the start. Total length = total of all edges + the shortest way to duplicate edges so every vertex becomes even. Find the odd vertices (at most four), list their pairings, choose the pairing of least total weight, and duplicate those edges. State the pairing before the total.

> **Worked example (original).** Edges AB 4, AC 2, BC 3, BD 5, CD 1 (total 15). Degrees: A 2, B 3, C 3, D 2 – the odd nodes are B and C, so duplicate BC (weight 3): the shortest closed route walking every edge at least once has length 15 + 3 = 18.

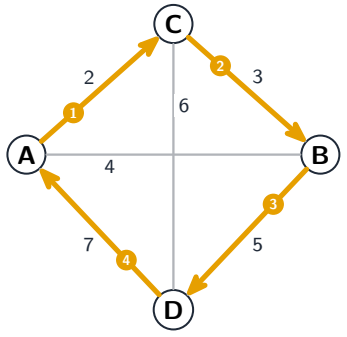
The travelling salesman problem

The **classical travelling salesman problem** 旅行商问题 tours every VERTEX exactly once and returns, on a complete graph with the triangle inequality. Bounds: the MST of the network doubled is an UPPER bound route (walk each MST edge twice); shortcuts give a better upper bound. Lower bounds come from MST-based arguments (e.g. the MST itself, or deleting a vertex and adding its two cheapest edges). The **nearest neighbour algorithm** 最近邻算法 builds a tour: from the

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start vertex repeatedly go to the nearest unvisited vertex, then return. Convert to a complete network of shortest distances first when the graph is not complete (via Dijkstra or inspection).

> **Worked example (original).** Complete network: AB 4, AC 2, AD 7, BC 3, BD 5, CD 6 (the triangle inequality holds throughout). Nearest neighbour from A: A–C (2), C–B (3), B–D (5), D–A (7): tour length 17 – an upper bound. The MST is AC + BC + BD = 10, so doubling it gives the crude upper bound 20, improved to 17 by shortcuts; a lower bound from deleting A (MST of B, C, D is $3 + 5 = 8$, plus A's two cheapest edges AC + AB $= 2 + 4 = 6$) is 14. The optimal tour lies between 14 and 17.



nearest neighbour from A:
A–C–B–D–A $= 2 + 3 + 5 + 7 = 17$
upper bound
delete A: MST of B, C, D $= 3 + 5 = 8$
plus AC + AB $= 2 + 4$: lower bound 14
$14 \leq$ optimal tour ≤ 17

Nearest neighbour gives an upper bound, the deleted-vertex MST a lower bound: the optimal tour lies between 14 and 17

Critical path analysis

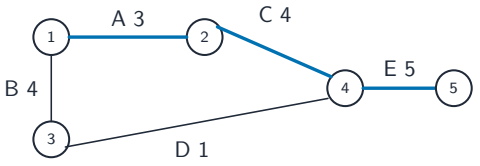
Model a project as an **activity-on-arc network** 活动在弧上的网络 from a precedence table; **dummies** 虚活动 (zero-weight arcs) resolve dependencies that shared endpoints cannot express. The **critical path** 关键路径 is the longest chain through the network: the project duration. Compute **earliest event times** 最早事件时间 (forward pass: the max of arrivals) and **latest event times** 最晚事件时间 (backward pass: the min of departures). An activity's **total float** 总时差 = latest finish − earliest start − duration; critical activities have zero float. **Gantt (cascade) charts** 甘特图 bar each activity from its earliest start with its float as a trailing space; scheduling allocates workers day by day.

> **Worked example (original).** Activities A (3 days), B (4), C (4 after A), D (1 after B), E (5 after C and D). Chain A–C–E = 12 days; chain B–D–E = 10 days: the critical path is A–C–E and the project takes 12 days; D's total float = 12 − (4 + 1 + 5) = 2 days.

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first-fit: (7,3), (5,4), (6,2) — three bins, none over capacity



event times 0; 3 / 4; 7; 12

critical path A–C–E = 12 days; D has float 2

Left: three first-fit bins of capacity 10 holding items (7,3), (5,4), (6,2). Right: an activity-on-arc network with events 1-5, activities A 3, B 4, C 4, D 1 and E 5, the critical arcs highlighted, and event times 0, 3/4, 7, 12.

Linear programming

Formulate: define the decision variables, write the objective (maximise/minimise) and every constraint as an inequality, including $x, y \geq 0$. Solve two-variable problems graphically: shade the feasible region, then use the **ruler method** 尺子法 (slide a ruler along the objective's gradient) or the **vertex method** 顶点法 (evaluate the objective at each vertex of the feasible region). Integer solutions: test integer points near the optimal vertex inside the region.

> **Worked example – linear programming (original).** Maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $2x + y \leq 14$, $x, y \geq 0$. The feasible region's vertices are $(0, 0)$, $(7, 0)$, $(4, 6)$, $(0, 10)$; evaluating P: 0, 21, 24, 20 – the maximum is $P = 24$ at $(4, 6)$, where both constraints are tight.

Checklist before you call this unit done

- Execute bubble and quick sorts showing every pass and pivot; bin pack by first-fit, first-fit decreasing and full-bin with placements shown.
- Find MSTs by Kruskal and by Prim (node and matrix forms); draw a network from a matrix and back.
- Run Dijkstra with labels in order of permanence; recover the route, not just the length.
- Solve route inspection: odd nodes, all pairings with weights, least duplication, total walk length.
- Bound the classical TSP: doubled MST and shortcuts for upper bounds, MST-deletion for lower bounds; run nearest neighbour on the complete shortest-distance network.
- Build activity networks with dummies; forward and backward passes for event times; total floats; identify the critical path; draw the Gantt chart and schedule.

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- Formulate linear programs fully; solve graphically by ruler or vertex methods; find best integer solutions.

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