

Mechanics 3

Pearson Edexcel · International A-Level · Further Mathematics

Mechanics 3: elastic strings, SHM, circles and centres of mass

Mechanics 3 (M3, paper WME03/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the final mechanics unit: P1–P4, M1 and M2 are assumed. The whole paper works symbolically – answers stay in m , g , a , L , r , and exact algebra is the marks. This reference covers acceleration as a function of x , elastic strings and springs, variable forces and simple harmonic motion, horizontal and vertical circular motion, and centres of mass by integration.

Must-know for M3 (not in the booklet):

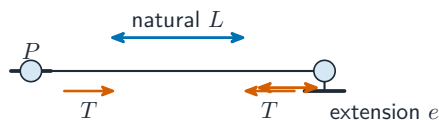
Family	Formulae
Hooke's law	tension $T = \frac{\lambda e}{L}$ (λ modulus, e extension, L natural length)
elastic energy	$E = \frac{\lambda e^2}{2L}$
SHM	$\ddot{x} = -\omega^2 x$, period $\frac{2\pi}{\omega}$, $v^2 = \omega^2(a^2 - x^2)$
circles	$a = r\omega^2 = \frac{v^2}{r}$ towards the centre; $v = r\omega$
COM	$\bar{x} = \frac{\int x y^2 dx}{\int y^2 dx}$ for a solid of revolution about x
kinematics	$a = v \frac{dv}{dx}$ when $a = f(x)$

Kinematics with acceleration as a function of x

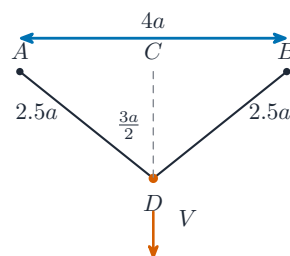
When acceleration depends on displacement, replace $\frac{dv}{dt}$ with $v \frac{dv}{dx}$ and integrate against x ; when it depends on time, separate $\frac{dx}{dt}$. The initial condition closes each constant – state it before substituting.

Worked example (WME03 June 2025 Q3). With $a = 4x - 2$ and at $t = 1$, $x = 0$, $v = 1$: (a) $v \frac{dv}{dx} = 4x - 2$ gives $\frac{1}{2}v^2 = 2x^2 - 2x + \frac{1}{2}$, so $v^2 = (2x - 1)^2$ and (positive direction) $v = 1 - 2x$. (b) Substituting $v = 1 - 2x$ into $a = \frac{dv}{dt}$ gives $\frac{dv}{dt} = -2v$ (or separate $\frac{dx}{dt} = 1 - 2x$): $\ln v = -2t + C$ with $v(1) = 1$ gives $v = e^{2-2t}$.

Elastic strings and springs



$$T = \frac{\lambda e}{L}, \text{ energy} = \frac{\lambda e^2}{2L}$$



each half: natural $\frac{3a}{2}$, so
at D energy $\frac{1}{2}mga$ total

Hooke's law : a light elastic string (or spring) of **natural length**

L and **modulus of elasticity** λ stretched by e exerts the tension $T = \frac{\lambda e}{L}$

– tension pulling inwards at both ends. The **elastic energy** stored is $\frac{\lambda e^2}{2L}$

is $\frac{\lambda e^2}{2L}$. **Work–energy principle** problems now carry three energy stores: kinetic, gravitational and elastic, plus any work done against resistance.

Worked example – a midpoint on a two-part string (WME03 June 2025 Q4).

A string with $\lambda = mg$, natural length $3a$, fixed between A and B ($AB = 4a$ horizontal); P hangs from the midpoint, projected downward from C (the midpoint of AB) at $\sqrt{3ag/2}$, and reaches D with $CD = \frac{3}{2}a$. Each half of the string has natural length $\frac{3a}{2}$: at C the half-length is $2a$ (extension $\frac{a}{2}$); at D it is $\sqrt{(2a)^2 + (\frac{3}{2}a)^2} = \frac{5a}{2}$ (extension a). Elastic energy gain per half: $\frac{mg}{2} \cdot \frac{3a}{2} (a^2 - \frac{a^2}{4}) = \frac{mg}{3a} \cdot \frac{3a^2}{4} = \frac{mga}{4}$; both halves: $\frac{1}{2}mga$ ✓. Then with resistance $\frac{1}{5}mg$ over $\frac{3a}{2}$ and the loss of gravitational energy $mg \cdot \frac{3a}{2}$:

$$\frac{3}{2}mga = \frac{1}{2}mV^2 - \frac{1}{2}m \cdot \frac{3ag}{2} + \frac{1}{2}mga + \frac{1}{5}mg \cdot \frac{3a}{2} \implies V^2 = \frac{29}{10}ag,$$

so $V = \sqrt{29ag/10}$.

Worked example – a hinged rod (WME03 June 2025 Q1). A uniform rod AB (mass m , length $2a$) hinged at A; an elastic string ($\lambda = 2mg$, natural length L) runs vertically from B to C, where AC is horizontal and $BC = 1.2a$ – the geometry puts B at $1.6a$ horizontally from A and $1.2a$ below it (a 3–4–5 triangle), so the rod's midpoint is $0.8a$ from the vertical through A. Moments about A: $T \cdot 1.6a = mg \cdot 0.8a$, so $T = \frac{1}{2}mg$; then Hooke's law gives $\frac{1}{2}mg = \frac{2mg}{L}(1.2a - L)$, i.e. $L = 0.96a = \frac{24}{25}a$.

Variable forces and simple harmonic motion

With variable force $F(x)$ in one dimension: $m \frac{dv}{dt} = F$ becomes $mv \frac{dv}{dx} = F(x)$; inverse-square gravitation appears as $F = \frac{k}{x^2}$. **Simple harmonic motion** (SHM)

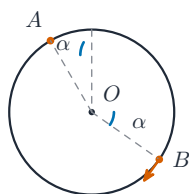
means $\ddot{x} = -\omega^2 x$ – the sign is the proof. Standard results (quotable): period $\frac{2\pi}{\omega}$, $v^2 = \omega^2(a^2 - x^2)$, $x = a \cos \omega t$ from an end, $a \sin \omega t$ from the centre. A particle between two elastic strings oscillates with SHM whenever the net force is proportional to displacement from the equilibrium point.

Worked example – SHM between two strings (WME03 June 2025 Q7). Strings from P to A ($\lambda = mg$, natural $3L$) and to B ($\lambda = 2mg$, natural $2L$), $AB = 7L$; equilibrium at E with $AE = 4.5L$ (each tension = $\frac{mg}{2}$ – the check that E is genuine). At displacement x from E towards B, the B-string shortens and the A-string lengthens:

$$m\ddot{x} = \frac{2mg}{2L} \left(\frac{L}{2} - x\right) - \frac{mg}{3L} \left(\frac{3L}{2} + x\right) = -\frac{4mg}{3L}x,$$

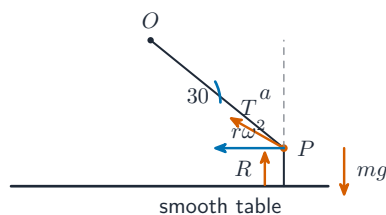
so P moves with SHM, $\omega^2 = \frac{4g}{3L}$ and the period is $\frac{2\pi}{\omega} = \pi\sqrt{\frac{3L}{g}}$. Released from C with $AC = 5L$ with **amplitude** $\frac{L}{2}$: $v_{\max} = \frac{L}{2}\omega = \sqrt{\frac{gL}{3}}$. The time with $|v| \leq \sqrt{\frac{gL}{12}} = \frac{v_{\max}}{2}$: $|\cos \omega t| \leq \frac{1}{2}$ holds for a third of every oscillation, so the slow time is $\frac{1}{3} \times \pi\sqrt{\frac{3L}{g}} = \pi\sqrt{\frac{L}{3g}}$.

Motion in a horizontal circle



heights above O: v
 $r \cos \alpha$, $r \sin \alpha$

leaves where $R = 0$: $mg \sin \alpha = \frac{mv^2}{r}$, so $\tan \alpha = \frac{2}{3}$



$$T \cos 30^\circ = m(a \cos 30^\circ)\omega^2; R = \frac{3}{4}mg$$

Angular speed ω (rad s^{-1}) with $v = r\omega$; the **centripetal acceleration** is $r\omega^2 = \frac{v^2}{r}$, directed towards the centre. Resolve towards the centre for the force equation and vertically for equilibrium – the **conical pendulum**, a particle on a table tethered by a string, and banked surfaces are the three standard contexts.

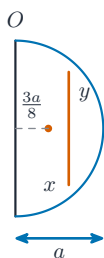
Worked example – a circle on a table (WME03 June 2025 Q2). P (mass m) on a smooth table, attached to O above by a string of length a making 30° with the table, moving in a circle with $\omega = \sqrt{\frac{g}{2a}}$. The circle's radius is $a \cos 30^\circ$. Towards the centre: $T \cos 30^\circ = m(a \cos 30^\circ)\omega^2 = ma \cos 30^\circ \cdot \frac{g}{2a}$, so $T = \frac{mg}{2}$. Vertically: $R = mg - T \sin 30^\circ = mg - \frac{mg}{4} = \frac{3mg}{4}$.

Motion in a vertical circle

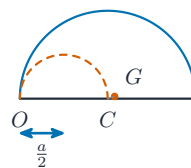
A **vertical circle** pairs energy with Newton's law towards the centre. Energy between levels ($\frac{1}{2}mv^2 = mg\Delta h$ plus any elastic terms) combines with Newton's law towards the centre; the leaving condition on a surface is $R = 0$, on a string $T = 0$. After leaving, the particle is a projectile: the horizontal component of velocity is conserved.

Worked example – leaving a sphere (WME03 June 2025 Q5). P released at A on a smooth sphere (radius r), OA at α above the vertical; it leaves at B, OB at α above the horizontal. (a) Energy A→B: the height of A above O is $r \cos \alpha$ and the height of B above O is $r \sin \alpha$, so P falls $r(\cos \alpha - \sin \alpha)$: $\frac{1}{2}mv^2 = mgr(\cos \alpha - \sin \alpha)$, giving $v^2 = 2gr(\cos \alpha - \sin \alpha)$. (b) At B with $R = 0$: $mg \sin \alpha = \frac{mv^2}{r}$ (the component of weight towards O); eliminating v^2 : $\sin \alpha = 2(\cos \alpha - \sin \alpha)$, so $\tan \alpha = \frac{2}{3}$. (c) Afterwards the horizontal component $v \sin \alpha$ is conserved; at the horizontal through O the speed satisfies $\frac{1}{2}mW^2 = mgr \cos \alpha$, and $\cos^2 \theta = \frac{v^2 \sin^2 \alpha}{W^2} = \frac{8}{78} = \frac{4}{39}$ – so $\cos \theta = \frac{2}{\sqrt{39}}$.

Centres of mass by integration



$$\bar{x} = \frac{\int_0^a x(a^2 - x^2) dx}{\int_0^a (a^2 - x^2) dx} = \frac{3a}{8}$$



masses 8:1:7; G at $\frac{45a}{112}$ along, $\frac{a}{14}$ off-axis; tilted rest $\tan \theta = \frac{8}{45}$

For a uniform **solid of revolution**, $\bar{x} = \frac{\int_0^a xy^2 dx}{\int_0^a y^2 dx}$ (weight thin discs by $\pi y^2 dx$). The **hemisphere** result must be derivable, not just quoted: with $y^2 = a^2 - x^2$, $\int_0^a x(a^2 - x^2) dx = \frac{a^4}{4}$ and $\int_0^a (a^2 - x^2) dx = \frac{2a^3}{3}$, giving $\bar{x} = \frac{3a}{8}$ from the plane face. **Composite bodies**: weight the parts by mass (radius-cubed for similar solids) and take moments; removed parts carry negative mass. Equilibrium of the tilted body: the COM sits vertically above the contact point – **equilibrium** of the tilted body.

Worked example – a carved hemisphere (WME03 June 2025 Q6). S is a solid hemisphere (radius a) with a radius- $\frac{a}{2}$ hemisphere removed, plane faces coplanar, centres O (big) and C with $OC = \frac{a}{2}$. Masses 8 : 1 : 7 (radii cubed); along the axis $8 \cdot \frac{3a}{8} - 1 \cdot \frac{3a}{16} = 7\bar{y}$, so the COM of S is $\frac{45a}{112}$ from the OC-line through O. Perpendicular

to OC (moments about the in-plane axis through O): $8 \cdot 0 - 1 \cdot \frac{a}{2} = 7\bar{x}$, so $\bar{x} = \frac{a}{14}$.
 Resting tilted on its curved surface, $\tan \theta = \frac{a/14}{45a/112} = \frac{8}{45}$.

Checklist before you call this unit done

- Convert $a = f(x)$ via $v dv/dx$ and close constants from a stated instant; finish the second integration by separation or by eliminating x .
- Use $T = \lambda e/L$ and elastic energy $\lambda e^2/2L$; for midpoint strings treat each half separately; run three-store work-energy with resistance over distance.
- Combine moments with Hooke's law for hinged rods in equilibrium (geometry first: horizontal/vertical offsets of the attachment points).
- Prove SHM by deriving $\ddot{x} = -\omega^2 x$ (equilibrium tensions first), then quote period, $v_{max} = a\omega$ and the sinusoidal time fractions.
- Resolve towards the centre ($r\omega^2$) and vertically for horizontal circles – table reactions, conical pendulums, banked surfaces.
- For vertical circles: energy between levels, $R = 0$ (or $T = 0$) at the leaving point, projectile continuation conserving the horizontal component.
- Derive the hemisphere COM by the disc integral; build composites with signed masses; close tilted equilibrium with the COM-above-contact condition.