Further Pure Mathematics 1

Pearson Edexcel · International A-Level · Further Mathematics

Further Pure Mathematics 1: complex numbers, matrices and proof

Further Pure Mathematics 1 (FP1, paper WFM01/01, 1 hour 30 minutes, 75 marks) is compulsory for International AS and A Level Further Mathematics and for International AS and A Level Pure Mathematics. P1–P2 are assumed and may be tested, including sign-change location of roots, rotating shapes about the origin, and dividing a cubic or quartic by a quadratic. This reference covers complex numbers, roots of quadratic equations, numerical methods, the parabola and rectangular hyperbola, matrix algebra, transformations, finite series and proof by induction.

Must-know for FP1 (not in the booklet):

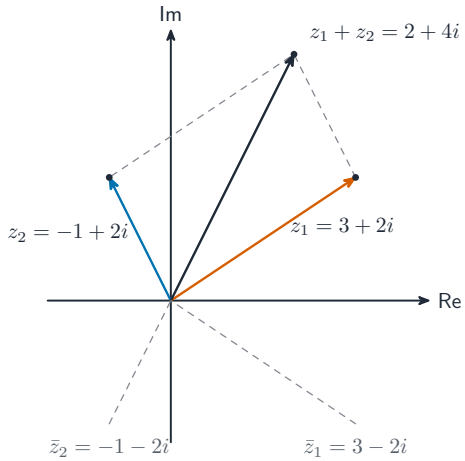
Family	Formulae
quadratic roots	for $ax^2+bx+c=0$ with roots α,β: $\alpha+\beta=-\dfrac{b}{a}$, $\quad\alpha\beta=\dfrac{c}{a}$
series	$\displaystyle\sum_{r=1}^{n} r=\tfrac{1}{2}n(n+1)$
complex numbers	$z=a+bi$, $\lvert z\rvert=\sqrt{a^2+b^2}$, $\bar{z}=a-bi$, $\lvert z_1z_2\rvert=\lvert z_1\rvert\lvert z_2\rvert$
conics	parabola $y^2=4ax$ (parametric $x=at^2$, $y=2at$); rectangular hyperbola $xy=c^2$ (parametric $x=ct$, $y=\frac{c}{t}$)
matrices	$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-bc$, $\quad\mathbf{M}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$, $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$

Complex numbers

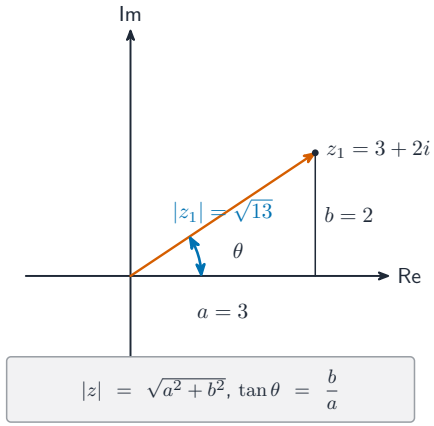
A **complex number** is $z=a+bi$ with $i^2=-1$; a is the **real part**, b the **imaginary part**. The **conjugate** $\bar{z}=a-bi$ reverses the sign of the imaginary part, and $z\bar{z}=a^2+b^2=|z|^2$ – the trick that makes a quotient real: $\dfrac{z_1}{z_2}=\dfrac{z_1\bar{z}_2}{|z_2|^2}$. Two complex numbers are equal only when both real and imaginary parts match. The polar form $z=r(\cos\theta+i\sin\theta)$ pairs the **modulus** r with the **argument** θ; multiplying complex numbers multiplies their moduli ($|z_1z_2|=|z_1||z_2|$).

Worked example – a real sum pins the imaginary part. $z_1=r(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6})$ with $r>0$, and z_2 has modulus 5. Since $|z_1z_2|=|z_1||z_2|=5r=15$, we get $r=3$. So $z_1=3(-\frac{\sqrt{3}}{2}-\frac{1}{2}i)=-\frac{3\sqrt{3}}{2}-\frac{3}{2}i$. If z_1+z_2 is real, its imaginary part is zero, so $z_2=a+\frac{3}{2}i$; then $a^2+\frac{9}{4}=25$ gives $a=\pm\frac{\sqrt{91}}{2}\approx\pm4.77$. Both $z_2=\frac{\sqrt{91}}{2}+\frac{3}{2}i$ and its reflection in the imaginary axis work.

On an **Argand diagram** (Argand diagram) plot $z=a+bi$ as the point (a,b): addition is vector addition, the conjugate reflects in the real axis, and $|z|$ is the distance from the origin.



conjugates reflect in the Re axis



Complex roots of polynomials

For a quadratic with real coefficients and negative discriminant, complete the square: $x^2-6x+13=(x-3)^2+4=0$ gives $x=3\pm2i$. The roots are a conjugate pair – and this survives degree: **if $f(z_1)=0$ for real coefficients, then $f(\bar{z}_1)=0$ too**. So a real cubic has either three real roots or one real root plus one conjugate pair, and $(z-z_1)(z-\bar{z}_1)=z^2-2\,\mathrm{Re}(z_1)z+|z_1|^2$ is a real quadratic factor. Divide it out to reach the rest.

Worked example – a quartic with a given complex root (WFM01 June 2025 Q7). $f(z)=Pz^4-36z^3+Qz^2+192z+68$ has root $3+5i$, so $3-5i$ is also a root and

$$z^2-6z+34 \text{ is a factor} \quad (\text{sum } 6,\ \text{product } 9+25=34).$$

Writing $f(z)=(z^2-6z+34)(Pz^2+bz+c)$ and comparing: $34c=68$ so $c=2$; $-6c+34b=192$ so $34b=204$, $b=6$; then $b-6P=-36$ gives $P=7$ and $c-6b+34P=Q$ gives $Q=204$. (Check against $f(3+5i)=0$: the real equation $644P+16Q+7772$-type balance holds since $644(7)+16(204)-7772=0$.) The other roots solve $7z^2+6z+2=0$:

$$z=\frac{-6\pm\sqrt{36-56}}{14}=\frac{-6\pm2i\sqrt{5}}{14}=\frac{-3\pm i\sqrt{5}}{7}.$$

Without a calculator: only $b^2 - 4ac < 0$ arithmetic and the conjugate fact are used.

Roots of quadratic equations

For $ax^2 + bx + c = 0$ with roots α, β: $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Symmetric expressions in α, β then need no individual roots: $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$, $\alpha^3 + \beta^3 \equiv (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$, and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$. To build a new quadratic, express the new **sum** and **product** in terms of the old ones; the equation is $x^2 - (\text{sum})x + \text{product} = 0$.

Worked example – new roots without solving (WFM01 June 2025 Q6). $f(x) = 3x^2 + kx - 5$ has roots α, β, so $\alpha\beta = -\frac{5}{3}$. If $\alpha + \beta = 9\alpha\beta$, then $-\frac{k}{3} = -15$, so $k = 45$ and $\alpha + \beta = -15$. For roots $\alpha^2 + \beta$ and $\alpha + \beta^2$:

$$\text{sum} = \alpha^2 + \beta^2 + \alpha + \beta = (\alpha+\beta)^2 - 2\alpha\beta + (\alpha+\beta) = 225 + \tfrac{10}{3} - 15 = \tfrac{640}{3},$$

$$\text{product} = (\alpha^3 + \beta^3) + \alpha\beta + (\alpha\beta)^2 = -3450 - \tfrac{5}{3} + \tfrac{25}{9} = -\tfrac{31040}{9}.$$

The new equation is $9x^2 - 1920x - 31040 = 0$ (multiply $x^2 - \frac{640}{3}x - \frac{31040}{9} = 0$ by 9).

Numerical solution of equations

Three iterative locators, all needing f continuous on the interval: **interval bisection** halves an interval carrying a sign change; **linear interpolation** chords from $(a, f(a))$ to $(b, f(b))$ and takes the chord's zero $\beta \approx a - \dfrac{(b-a)f(a)}{f(b) - f(a)}$; the **Newton–Raphson method** - (formula given in the booklet) iterates $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Show every evaluation – "calculator-only" answers score nothing.

Worked example – bisection and interpolation (WFM01 June 2025 Q3). (i) $f(x) = x^2 + 5 - 8^{5x}$ has a single root α in $[0, 1]$: $f(0) = 4 > 0$, $f(0.5) = 5.25 - 8^{2.5} = -175.77 < 0$, so halve: $f(0.25) = 5.0625 - 8^{1.25} = -8.392 < 0$. The sign change sits in $[0, 0.25]$. (ii) $g(x) = 3^{\sin x} - 3\cos x$ (radians): $g(4) = 2.396 > 0$ and $g(5) = -0.502 < 0$, and g is continuous, so a root β lies in $[4, 5]$. Interpolating once: $\beta \approx 4 + \dfrac{2.396}{2.396 + 0.502} = 4.827$ (4 s.f.).

Newton–Raphson (original example). Solve $x^3 - 2x - 5 = 0$ from $x_0 = 2$: $f(2) = -1$, $f'(x) = 3x^2 - 2$ so $f'(2) = 10$ and $x_1 = 2 - \frac{-1}{10} = 2.1$; then $f(2.1) = 0.061$, $f'(2.1) = 11.23$, $x_2 = 2.0946$. The iterates home in on the true root $2.0946\ldots$ (check: $2.0946^3 - 2(2.0946) - 5 \approx +0.0005$).

Parabola and rectangular hyperbola

The **parabola** $y^2 = 4ax$ has parametric form $x = at^2$, $y = 2at$; the **rectangular hyperbola** $xy = c^2$ has $x = ct$, $y = \frac{c}{t}$. Tangents and normals come from differentiating the Cartesian equation (parametric differentiation is not required): $y^2 = 4ax$ gives $\frac{dy}{dx} = \frac{2a}{y} = \frac{1}{t}$ at parameter t, and $xy = c^2$ gives $y = \frac{c^2}{x}$ so $\frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{1}{t^2}$ at $x = ct$.

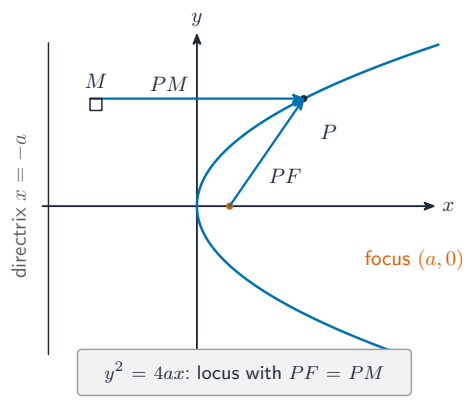
Worked example – normal to a hyperbola (WFM01 June 2025 Q4). For $xy = 81$ at $P(9t, \frac{9}{t})$: $\frac{dy}{dx} = -\frac{81}{x^2} = -\frac{1}{t^2}$, so the normal has gradient t^2:

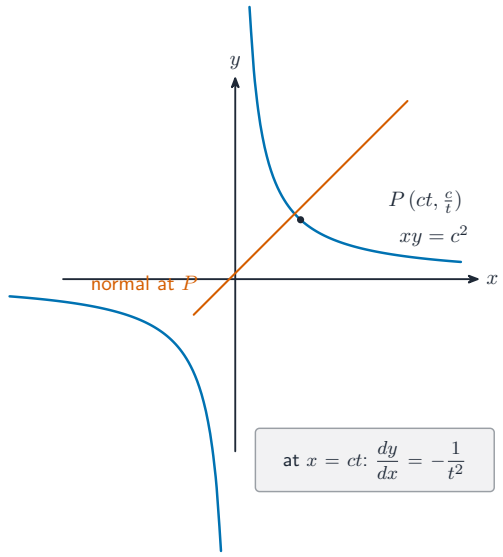
$$y - \tfrac{9}{t} = t^2(x - 9t) \implies ty = t^3x + 9(1 - t^4).$$

At the y-axis, $A = (0, \frac{9(1-t^4)}{t})$. With $t = \frac{1}{3}$: $P = (3, 27)$ and $A = (0, \frac{80}{3})$, so triangle OPA has area $\frac{1}{2} \times 3 \times \frac{80}{3} = 40$.

The parabola is the locus of points equidistant from the **focus** $(a, 0)$ and the **directrix** $x = -a$. Distances, not algebra, unlock the geometry questions.

Worked example – focus-directrix distances (WFM01 June 2025 Q9). P lies on the parabola 5 units from the focus, so P is also 5 units from the directrix ($PF = PM = 5$, with PM perpendicular to the directrix). Points Q, R on the directrix satisfy $PQ = PR = 10$. Each right triangle PQM gives $QM = \sqrt{10^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$, and M lies between Q and R, so $QR = 10\sqrt{3}$.





Matrix algebra

Matrices add entry-wise (same shape), scale entry-wise, and multiply row-by-column ($(\mathbf{AB})_{ik} = \sum_j A_{ij}B_{jk}$; $\mathbf{AB} \neq \mathbf{BA}$ in general). For 2×2 matrices $\det \mathbf{M} = ad - bc$: zero means **singular** (no inverse); otherwise $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$. Inverses reverse order: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ – undo the last transformation first.

Worked example – an inverse that must stay consistent (WFM01 June 2025 Q1). $\mathbf{M} = \begin{pmatrix} 1 & a \\ 3 & -5 \end{pmatrix}$ with $a \neq -\frac{5}{3}$:

$$\mathbf{M}^{-1} = \frac{1}{-(5+3a)}\begin{pmatrix} -5 & -a \\ -3 & 1 \end{pmatrix}.$$

Given $\mathbf{M}^{-1} = 2\mathbf{M} + 8\mathbf{I} = \begin{pmatrix} 10 & 2a \\ 6 & -2 \end{pmatrix}$, compare entries: $\dfrac{-3}{-(5+3a)} = 6$ gives $5 + 3a = \frac{1}{2}$, and every other entry confirms it, so $a = -\frac{3}{2}$.

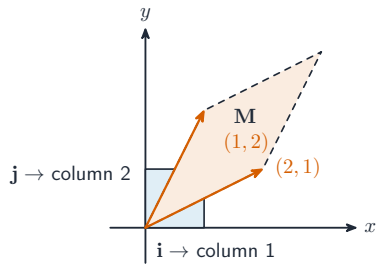
Transformations using matrices

A 2×2 matrix $\mathbf{M}$ transforms column vectors by $\mathbf{y} = \mathbf{Mx}$; its **columns are the images of i and j** – read any transformation straight off the matrix. The standard results:

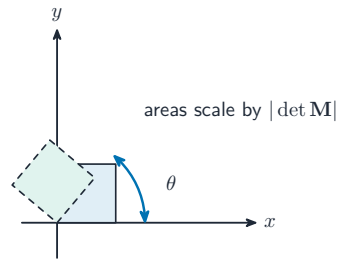
Transformation	Matrix
reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
reflection in $y = x$ / $y = -x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ / $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
rotation, angle θ anticlockwise about O	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
stretch, factor k parallel to the x-axis	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ (swap for the y-axis)
enlargement, scale factor k about O	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

For combinations, **AB** means "apply **B** first, then **A**". The inverse transformation undoes it, and $|\det \mathbf{M}|$ is the **area scale factor**: areas multiply by the determinant (negative determinants add a reflection).

Worked example – describing and combining (WFM01 June 2025 Q2). $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ maps $\mathbf{i} \to (0,-1)^T$ and $\mathbf{j} \to (1,0)^T$: a quarter-turn clockwise (270° anticlockwise) about O. $\mathbf{Q} = 5\mathbf{I}$ is an enlargement, scale factor 5, centre O. Reflection in $y = -x$ is $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, so $A = \mathbf{R}(4,3)^T = (-3,-4)$. If $\mathbf{Q}$ follows $\mathbf{P}$ and maps B to A, then $B = \mathbf{P}^{-1}\mathbf{Q}^{-1}A = \mathbf{P}^{-1} \cdot \frac{1}{5}(-3,-4)^T = \frac{1}{5}(4,-3)^T = (0.8,-0.6)$ – $\mathbf{P}$'s own inverse is its transpose, so this is arithmetic only.



$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: columns are the images of $\mathbf{i}, \mathbf{j}$



rotation by θ: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Series

With $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (and $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$ from the booklet), any polynomial summand splits into standard pieces – including sums to an even limit $2n$ (substitute $k = 2n$ into the standard formulae) and shifted sums like $\sum (r+1)^2$.

Worked example – a sum to 2n (WFM01 June 2025 Q8a). Show $\sum_{r=1}^{2n}(2r^2 - 1) = \frac{4}{3}n(n+1)(an+b)$:

$$2\sum_{1}^{2n} r^2 - 2n = 2 \cdot \frac{(2n)(2n+1)(4n+1)}{6} - 2n = \frac{2n(2n+1)(4n+1) - 6n}{3} = \frac{2n(8n^2+6n-2)}{3} = \frac{4n(4n^2+3n-1)}{3},$$

and $4n^2 + 3n - 1 = (n+1)(4n-1)$, so the sum is $\frac{4}{3}n(n+1)(4n-1)$: $a = 4$, $b = -1$. (Check $n = 1$: $1 + 7 = 8$ and $\frac{4}{3} \cdot 1 \cdot 2 \cdot 3 = 8$.)

Proof by mathematical induction

Induction proves a statement $P(n)$ for all positive integers in three moves: verify $P(1)$; assume $P(k)$; deduce $P(k+1)$ from it – then close with "true for $n = 1$, and truth for $n = k$ implies truth for $n = k+1$, so by induction true for all

positive integers n". The closing sentence is part of the proof. Four assessed families: summation formulae, divisibility, sequences given by a recurrence, and matrix powers.

Worked example – a summation (WFM01 June 2025 Q8b). Prove $\sum_{r=1}^{n} r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$. For $n=1$: LHS $= 1 \cdot 1^2 = 1$ and RHS $= \frac{1}{4} \cdot 1 \cdot 2 \cdot 2 = 1$ – true. Assume true for $n=k$; the $(k+1)$th term is $(k+1)(3k+1)^2$, so

$$\sum_{r=1}^{k+1} r(3r-2)^2 = \tfrac{1}{4}k^2(k+1)(9k-7) + (k+1)(3k+1)^2 = \tfrac{1}{4}(k+1)\big[k^2(9k-7) + 4(3k+1)^2\big].$$

Expand the bracket: $9k^3 - 7k^2 + 36k^2 + 24k + 4 = 9k^3 + 29k^2 + 24k + 4$. The target for $n = k+1$ is $\frac{1}{4}(k+1)^2(k+2)(9k+2)$, and its bracket $(k+1)(k+2)(9k+2)$ expands as $(k^2+3k+2)(9k+2) = 9k^3 + 29k^2 + 24k + 4$ – identical. So the statement holds for $n = k+1$, and by induction for all positive integers n.

Divisibility (original example). Prove $n^3 + 2n$ is divisible by 3. For $n=1$: 3. Assume $k^3+2k = 3m$; then $(k+1)^3+2(k+1) = (k^3+2k)+3k^2+3k+3 = 3(m+k^2+k+1)$, still a multiple of 3. By induction, true for all n.

Matrix powers (original example). Prove $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. True for $n = 1$. If true for k, then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$, completing the induction.

Checklist before you call this unit done

- Work with $a+bi$ and $r(\cos\theta + i\sin\theta)$: modulus, argument, conjugate; quotients made real.
- Use conjugate pairs to factor cubics and quartics with real coefficients; find all roots without a calculator.
- Convert symmetric expressions via $\alpha+\beta$ and $\alpha\beta$; form new quadratics for transformed roots.
- Locate roots by bisection and linear interpolation with every evaluation shown; iterate Newton–Raphson from a given start.
- Use $y^2 = 4ax$ and $xy = c^2$ parametrics; tangents/normals via Cartesian differentiation; focus-directrix distances.
- Invert 2×2 matrices (and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$); solve matrix equations entry by entry.
- Identify reflections, rotations, stretches, enlargements from matrices; combine in the right order; use $|\det|$ as the area scale factor.
- Sum series to n and $2n$ with the standard formulae; prove series, divisibility, recurrence and matrix results by induction with a closing statement.