

Solutions

2.1 With $y = \frac{R}{x}h$: $\bar{x} = \frac{\int_0^h x \cdot \frac{R^2 x^2}{h^2} dx}{\int_0^h \frac{R^2 x^2}{h^2} dx} = \frac{h^4/4}{h^3/3} = \frac{3h}{4}$ from the vertex, i.e. $\frac{h}{4}$ from the base ✓.

2.2 Masses 8 : 1 : 7: $\bar{x} = \frac{8 \cdot \frac{3a}{8} - 1 \cdot \frac{3a}{16}}{7} = \frac{3a - \frac{3a}{16}}{7} = \frac{45a}{16 \cdot 7} = \frac{45a}{112}$ from the common plane face.

2.3 Rolling the cylinder about its axis keeps the centre of mass (on the axis) at a constant height, so every such position is neutral equilibrium. The axis is a distance a (the radius) above the ground, so the centre of mass is at height a .

3.1 (a) Slicing perpendicular to the x-axis (measured from O along the symmetry axis): $\bar{x} = \frac{\int_0^a x(a^2 - x^2) dx}{\int_0^a (a^2 - x^2) dx}$. Numerator: $[\frac{a^2 x^2}{2} - \frac{x^4}{4}]_0^a = \frac{a^4}{2} - \frac{a^4}{4} = \frac{a^4}{4}$. Denominator: $[a^2 x - \frac{x^3}{3}]_0^a = a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$ (so the volume is $\frac{2}{3}\pi a^3$ ✓). Hence $\bar{x} = \frac{a^4/4}{2a^3/3} = \frac{3a}{8}$ ✓. (b) The small hemisphere has radius $\frac{a}{2}$, so its mass ratio is $\frac{(a/2)^3}{a^3} = \frac{1}{8}$ and its COM is $\frac{3}{8} \cdot \frac{a}{2} = \frac{3a}{16}$ from O along OC. With masses 8 : 1 : 7: $8 \cdot \frac{3a}{8} - 1 \cdot \frac{3a}{16} = 7\bar{y}$, i.e. $3a - \frac{3a}{16} = \frac{45a}{16} = 7\bar{y}$, so $\bar{y} = \frac{45a}{112}$ ✓. (c) Take the moment axis in the plane face, through O, perpendicular to OC. The big hemisphere's COM sits $\frac{3a}{8}$ above O measured perpendicular to the face, directly above O, so its in-plane offset from the axis is 0. The small hemisphere's COM sits above C, whose in-plane offset along OC is $\frac{a}{2}$ (its height $\frac{3a}{16}$ above the face is parallel to the axis and contributes nothing). Moments about the axis: $8 \cdot 0 - 1 \cdot \frac{a}{2} = 7\bar{x}$, so the COM of S is $\frac{a}{14}$ off the OC-line on the side away from C. Resting on its curved surface, the COM lies vertically above the contact point, so $\tan \theta = \frac{a/14}{45a/112} = \frac{8}{45}$ ✓.

Local review guidance: the disc integral with the correct limits and y^2 ; composite moments with cubed mass ratios; (dM1) the tilted rest closing with the COM vertically above the contact. This is authored classroom guidance.