

KEY WORDS integration by parts • completing the square

Integrating hyperbolic and inverse functions

Objective

Integrate hyperbolic expressions, use the four standard surd/rational forms, complete the square under surds, and integrate inverse functions by parts (FP3 4.1–4.4).

1 Key knowledge and worked examples

Standard forms. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arsinh} \frac{x}{a} + c$, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arcosh} \frac{x}{a} + c$,
 $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$, $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$. Complete the square first when the quadratic has a linear term. For $\int (\text{inverse function}) dx$, integrate by parts with the inverse function as u and $1 \cdot dx$ as dv .

Worked example – both techniques (WFM03 June 2025 Q2). (i) $4x^2 + 8x + 9 = 4((x+1)^2 + \frac{5}{4})$, so $\int \frac{dx}{\sqrt{4x^2 + 8x + 9}} = \frac{1}{2} \operatorname{arsinh} \frac{2x+2}{\sqrt{5}} + c$. (ii) $\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \int \frac{3x \, dx}{\sqrt{9x^2 - 1}} = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c$.

2 Practice

2.1 Find $\int \sinh^3 x \cosh x \, dx$. [3]

2.2 Evaluate $\int_0^1 \frac{dx}{x^2 + 4}$. [4]

2.3 Find $\int \frac{dx}{x^2 - 9}$. [4]

2.4 Find $\int \frac{dx}{\sqrt{x^2 + 9}}$. [3]

3 Exam-style questions

3.1 (*Edexcel IAL WFM03 June 2025 Q2.*) Determine

(i) $\int \frac{dx}{\sqrt{4x^2 + 8x + 9}}$

(ii) $\int \operatorname{arcosh} 3x \, dx.$ [7]

3.2 (*Original.*) Evaluate $\int_0^1 \operatorname{arsinh} x \, dx$, giving your answer in exact form. [4]
