2.2 The method of differences

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS method of differences • partial fractions • closed form

The method of differences

Objective

Sum finite series by splitting the summand (partial fractions or factorial identities) so that consecutive terms cancel, writing the $r = 1, 2, n-1, n$ lines that display the collapse, and close follow-ups by algebra (FP2 2.1).

1 Key knowledge and worked examples

The mechanism. If $u_r = v_r - v_{r+1}$, then $\sum_{r=1}^{n} u_r = v_1 - v_{n+1}$: each middle term appears once positive and once negative. Show the collapse explicitly – write the first two and last two terms and strike through what cancels. Partial fractions supply the split for rational summands; $(r+1)! - r! = r \cdot r!$ supplies it for factorials.

Worked example – telescoping to a closed form (WFM02 June 2025 Q5). $\dfrac{1}{(r+1)(r+2)} = \dfrac{1}{r+1} - \dfrac{1}{r+2}$, so $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$ (everything from $\frac{1}{3}$ to $\frac{1}{n+1}$ cancels). Check $n = 1$: both sides $\frac{1}{6}$. A follow-up like "smallest n with sum > 0.49" is then algebra: $\dfrac{n}{2(n+2)} > 0.49 \Rightarrow n > 98$, so $n = 99$.

2 Practice

2.1 Show that $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$. [2]

2.2 Show that $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$. [4]

2.3 Show that $\displaystyle\sum_{r=1}^{n} r \cdot r! = (n+1)! - 1$. [3]

3 Exam-style questions

3.1 *(Original.)* (a) Show that $\dfrac{1}{r(r+1)(r+2)} = \dfrac{1}{2}\left[\dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}\right]$. (b) Hence find, in its simplest form, $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [6]

3.2 *(Adapted from Edexcel IAL WFM02 June 2025 Q5.)* (a) Express $\dfrac{1}{(r+1)(r+2)}$ in partial fractions. (b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$. (c) Determine the smallest value of n for which the sum exceeds 0.49. [10]