

6.1 Kinematics and elastic strings

Name: _____ Class: _____ Date: _____ Total: 25 marks

KEY WORDS Hooke’s law • elastic energy

Kinematics and elastic strings

Objective

Solve $a = f(x)$ problems via $v \, dv/dx$ with constants closed by initial conditions, and apply Hooke’s law and elastic energy to strings and springs in equilibrium and in work-energy problems (objectives 1.1, 2.1, 2.2).

1 Key knowledge and worked examples

Two toolkits. $a = f(x)$: replace $\frac{dv}{dt}$ by $v \frac{dv}{dx}$, integrate, close the constant from a stated instant. Elastic string of natural length L , modulus λ , extension e : tension $\frac{\lambda e}{L}$, stored energy $\frac{\lambda e^2}{2L}$; each half of a cut string has half the natural length and half the extension.

Worked example – both (WME03 June 2025 Q3, Q4). (i) $a = 4x - 2$, at $t = 1$: $x = 0$, $v = 1$: $v \frac{dv}{dx} = 4x - 2$ gives $v^2 = (2x - 1)^2$ so $v = 1 - 2x$; then $\frac{dv}{dt} = -2v$ gives $v = e^{2-2t}$. (ii) A string ($\lambda = mg$, natural $3a$) spans $AB = 4a$; P at the midpoint drops $\frac{3a}{2}$: each half has natural length $\frac{3a}{2}$ and length $\frac{5a}{2}$ at D, so the elastic energy gain is $\frac{mg}{3a} \cdot \frac{3a^2}{4} = \frac{mga}{4}$ per half – $\frac{1}{2}mga$ in total – and the work-energy equation with resistance $\frac{1}{5}mg$ gives $V = \sqrt{29ag/10}$.

2 Practice

2.1 A particle has $a = 6x$ and passes $x = 1$ with $v = 2$. Find v^2 in terms of x . [3]

2.2 An elastic string has natural length 1.2 m and modulus 24 N. Find the tension when the extension is 0.3 m, and the energy stored. [3]

2.3 A spring of natural length L and modulus λ hangs vertically with a mass m in equilibrium. Show that the extension is mgL/λ . [2]

3 Exam-style questions

3.1 (*Edexcel IAL WME03 June 2025 Q3.*) A particle P moves along the x-axis. At time t seconds ($t \geq 1$) the displacement of P from O is x metres (positive x direction), where $x < \frac{1}{2}$; the velocity is $v \text{ ms}^{-1}$ and the acceleration $a \text{ ms}^{-2}$ (both positive x direction). Given $a = 4x - 2$ and that at $t = 1$, $x = 0$ and $v = 1$: (a) find v in terms of x ; (b) find v in terms of t . [9]

3.2 (*Edexcel IAL WME03 June 2025 Q4.*) A light elastic string of natural length $3a$ has modulus of elasticity mg ; its ends are attached to fixed points A and B with AB horizontal and $AB = 4a$. A particle P of mass m is attached to the midpoint of the string and held at rest at C, the midpoint of AB. P is projected vertically downwards from C with speed $\sqrt{3ag/2}$ and reaches D, where $CD = \frac{3}{2}a$, with speed V . (a) Show that the elastic energy stored in the string increases by $\frac{1}{2}mga$ as P moves from C to D. (b) Given the air resistance is a constant force of magnitude $\frac{1}{5}mg$, use the work-energy principle to find V in terms of a and g . [8]

Solutions

2.1 $v \frac{dv}{dx} = 6x$: $\frac{1}{2}v^2 = 3x^2 + C$; at $x = 1$, $v = 2$ gives $C = -1$: $v^2 = 6x^2 - 2$.

2.2 $T = \frac{24 \times 0.3}{1.2} = 6$ N; energy $= \frac{24 \times 0.3^2}{2 \times 1.2} = 0.9$ J.

2.3 Equilibrium: $mg = T = \frac{\lambda e}{L}$, so $e = \frac{mgL}{\lambda}$ ✓.

3.1 (a) $v \frac{dv}{dx} = 4x - 2$: $\frac{1}{2}v^2 = 2x^2 - 2x + C$; at $(x, v) = (0, 1)$, $C = \frac{1}{2}$: $v^2 = 4x^2 - 4x + 1 = (2x - 1)^2$. Since $x < \frac{1}{2}$ and v is positive, $v = 1 - 2x$. (b) From (a), $a = \frac{dv}{dt} = -2v$ (differentiate $1 - 2x$): $\int \frac{dv}{v} = -2 \int dt$ gives $\ln v = -2t + C$; $v(1) = 1$ gives $C = 2$, so $v = e^{2-2t}$. (Or separate $\frac{dx}{1-2x} = dt$: $x = \frac{1}{2}(1 - e^{2-2t})$, then differentiate.)

3.2 (a) Each half of the string has natural length $\frac{3a}{2}$. At C the half-length is $2a$ (extension $\frac{a}{2}$); at D it is $\sqrt{(2a)^2 + (\frac{3}{2}a)^2} = \frac{5a}{2}$ (extension a). Energy per half: $\frac{mg}{2 \cdot \frac{3a}{2}}(a^2 - \frac{a^2}{4}) = \frac{mg}{3a} \cdot \frac{3a^2}{4} = \frac{mga}{4}$; both halves: $\frac{1}{2}mga$ ✓. (b) Loss of GPE $mg \cdot \frac{3a}{2}$; gain of KE $\frac{1}{2}mV^2 - \frac{1}{2}m \cdot \frac{3ag}{2}$; gain of elastic energy $\frac{1}{2}mga$; work against resistance $\frac{1}{5}mg \cdot \frac{3a}{2}$. So $\frac{3}{2}mga = \frac{1}{2}mV^2 - \frac{3}{4}mga + \frac{1}{2}mga + \frac{3}{10}mga$, i.e. $\frac{1}{2}V^2 = \frac{29}{20}ag$: $V = \sqrt{\frac{29}{10}ag}$.

Local review guidance: the $v dv/dx$ substitution before integrating in x ; half-strings treated with their own natural lengths; work-energy with every store present and signs stated. This is authored classroom guidance.