

3.4 Integrating hyperbolic and inverse functions

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS integration by parts • completing the square

Integrating hyperbolic and inverse functions

Objective

Integrate hyperbolic expressions, use the four standard surd/rational forms, complete the square under surds, and integrate inverse functions by parts (FP3 4.1–4.4).

1 Key knowledge and worked examples

Standard forms. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arsinh} \frac{x}{a} + c$, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arcosh} \frac{x}{a} + c$,
 $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$, $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + c$. Complete the square first when the quadratic has a linear term. For $\int (\text{inverse function}) dx$, integrate by parts with the inverse function as u and $1 \cdot dx$ as dv .

Worked example – both techniques (WFM03 June 2025 Q2). (i) $4x^2 + 8x + 9 = 4\left((x + 1)^2 + \frac{5}{4}\right)$, so $\int \frac{dx}{\sqrt{4x^2 + 8x + 9}} = \frac{1}{2} \operatorname{arsinh} \frac{2x + 2}{\sqrt{5}} + c$. (ii)
 $\int \operatorname{arcosh} 3x dx = x \operatorname{arcosh} 3x - \int \frac{3x dx}{\sqrt{9x^2 - 1}} = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c$.

2 Practice

2.1 Find $\int \sinh^3 x \cosh x dx$. [3]

2.2 Evaluate $\int_0^1 \frac{dx}{x^2 + 4}$. [4]

2.3 Find $\int \frac{dx}{x^2 - 9}$. [4]

2.4 Find $\int \frac{dx}{\sqrt{x^2 + 9}}$. [3]

3 Exam-style questions

3.1 (*Edexcel IAL WFM03 June 2025 Q2.*) Determine

(i) $\int \frac{dx}{\sqrt{4x^2 + 8x + 9}}$

(ii) $\int \operatorname{arcosh} 3x dx$. [7]

3.2 (*Original.*) Evaluate $\int_0^1 \operatorname{arsinh} x \, dx$, giving your answer in exact form. [4]

Solutions

2.1 With $u = \sinh x$: $\frac{1}{4} \sinh^4 x + c$.

2.2 $\left[\frac{1}{2} \arctan \frac{x}{2}\right]_0^1 = \frac{1}{2} \arctan \frac{1}{2}$.

2.3 $\frac{1}{2 \cdot 3} \ln \left| \frac{x-3}{x+3} \right| + c = \frac{1}{6} \ln \left| \frac{x-3}{x+3} \right| + c$.

2.4 $\operatorname{arsinh} \frac{x}{3} + c \quad (= \ln(x + \sqrt{x^2 + 9}) + c')$.

3.1 (i) $4x^2 + 8x + 9 = (2x + 2)^2 + 5$, so with $u = 2x + 2$, $dx = \frac{du}{2}$: $\frac{1}{2} \int \frac{du}{\sqrt{u^2 + 5}} = \frac{1}{2} \operatorname{arsinh} \frac{2x+2}{\sqrt{5}} + c$. (ii) By parts, $u = \operatorname{arcosh} 3x$, $dv = dx$: $x \operatorname{arcosh} 3x - \int x \cdot \frac{3}{\sqrt{9x^2 - 1}} dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c$ (because $\frac{d}{dx} \sqrt{9x^2 - 1} = \frac{9x}{\sqrt{9x^2 - 1}}$).

3.2 By parts: $[x \operatorname{arsinh} x]_0^1 - \int_0^1 \frac{x dx}{\sqrt{1+x^2}} = \ln(1+\sqrt{2}) - \left[\sqrt{1+x^2}\right]_0^1 = \ln(1+\sqrt{2}) - \sqrt{2} + 1$.

Local review guidance: square completed correctly under the surd (or linear substitution identified); by parts with the inverse function chosen as u ; exact forms with the constant of integration on indefinite integrals. This is authored classroom guidance.