

3.3 Differentiating hyperbolic and inverse functions

Name: _____ Class: _____ Date: _____

Total: 25 marks

KEY WORDS inverse function • chain rule

Differentiating hyperbolic and inverse functions

Objective

Differentiate hyperbolic expressions by the chain, product and quotient rules, and inverse trigonometric/hyperbolic functions by implicit inversion, respecting domains (FP3 3.1–3.2).

1 Key knowledge and worked examples

Derivatives. $\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$, $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$;
 $\frac{d}{dx} \operatorname{arsinh} x = \frac{1}{\sqrt{1+x^2}}$, $\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx} \operatorname{artanh} x = \frac{1}{1-x^2}$, $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$. Where a domain like $x > 0$ is given, use it to resolve $\sqrt{x^2} = x$.

Worked example – inverse derivatives on a restricted domain (WFM03 June 2025 Q4). For $y = \operatorname{arsinh} x + \operatorname{arsinh} \frac{1}{x}$, $x > 0$: $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}} = \frac{x-1}{x\sqrt{1+x^2}}$. The stationary point is $x = 1$, where $y = 2 \operatorname{arsinh} 1 = 2 \ln(1 + \sqrt{2})$.

2 Practice

2.1 Differentiate $x \sinh 2x$.

[4]

2.2 Differentiate $\cosh(3x^2 + 1)$. [3]

2.3 Differentiate $x^2 \operatorname{artanh} x$. [3]

2.4 Show that $\frac{d}{dx}(e^x \cosh x) = e^{2x}$. [3]

3 Exam-style questions

3.1 (*Edexcel IAL WFM03 June 2025 Q4.*) $y = \operatorname{arsinh} x + \operatorname{arsinh} \frac{1}{x}$, $x > 0$. (a) Show that $\frac{dy}{dx} = \frac{x-1}{x\sqrt{1+x^2}}$. (b) Hence determine the exact value of y for which $\frac{dy}{dx} = 0$, giving your answer as a simplified natural logarithm. [6]

3.2 (*Original.*) (a) Differentiate $\operatorname{arcosh} 2x$. (b) Find the equation of the tangent to $y = \operatorname{arcosh} 2x$ at the point where $x = 1$, giving exact values. [6]

Solutions

2.1 Product rule: $\sinh 2x + 2x \cosh 2x$.

2.2 Chain rule: $6x \sinh(3x^2 + 1)$.

2.3 $2x \operatorname{artanh} x + \frac{x^2}{1 - x^2}$.

2.4 $\frac{d}{dx}(e^x \cosh x) = e^x \cosh x + e^x \sinh x = e^x(\cosh x + \sinh x) = e^x \cdot e^x = e^{2x}$, since $\cosh x + \sinh x \equiv e^x$.

3.1 (a) $\frac{d}{dx} \operatorname{arsinh} x = \frac{1}{\sqrt{1+x^2}}$ and, by the chain rule, $\frac{d}{dx} \operatorname{arsinh} \frac{1}{x} = \frac{-1/x^2}{\sqrt{1+1/x^2}}$. For $x > 0$, $\sqrt{1+1/x^2} = \frac{\sqrt{1+x^2}}{x}$, so the second term is $-\frac{1}{x\sqrt{1+x^2}}$ and $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \left(1 - \frac{1}{x}\right) = \frac{x-1}{x\sqrt{1+x^2}}$. (b) $\frac{dy}{dx} = 0$ exactly at $x = 1$: $y = \operatorname{arsinh} 1 + \operatorname{arsinh} 1 = 2 \ln(1 + \sqrt{2})$.

3.2 (a) $\frac{d}{dx} \operatorname{arcosh} 2x = \frac{2}{\sqrt{4x^2-1}}$ (chain rule on $\sqrt{(2x)^2-1}$). (b) At $x = 1$: $y = \operatorname{arcosh} 2 = \ln(2 + \sqrt{3})$ and the gradient is $\frac{2}{\sqrt{3}}$. Tangent: $y - \ln(2 + \sqrt{3}) = \frac{2}{\sqrt{3}}(x - 1)$.

Local review guidance: correct inverse derivative including the inner-function factor; the $x > 0$ simplification of $\sqrt{x^2}$ credited only when justified; exact values as logarithms or surds, never decimals. This is authored classroom guidance.