

3.2 The ellipse and the hyperbola

Name: _____ Class: _____ Date: _____ Total: 25 marks

KEY WORDS ellipse • hyperbola • eccentricity • directrix

The ellipse and the hyperbola

Objective

Use the standard equations and parametric forms of the ellipse and hyperbola, their eccentricity/focus/directrix relations, and tangent conditions (FP3 2.1–2.4).

1 Key knowledge and worked examples

Standard facts. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($x = a \cos t$, $y = b \sin t$): $b^2 = a^2(1 - e^2)$, foci $(\pm ae, 0)$, directrices $x = \pm \frac{a}{e}$. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ($x = a \sec t$, $y = b \tan t$; right branch $x = a \cosh t$, $y = b \sinh t$): $b^2 = a^2(e^2 - 1)$. Tangent condition for $y = mx + c$: $c^2 = a^2m^2 + b^2$ (ellipse) or $c^2 = a^2m^2 - b^2$ (hyperbola). Focus property: $PS = e \cdot PN$ at every point.

Worked example – eccentricity, tangent, intercepts (WFM03 June 2025 Q9, condensed). For $\frac{x^2}{64} - \frac{y^2}{49} = 1$: $49 = 64(e^2 - 1)$ gives $e = \frac{\sqrt{113}}{8}$, foci $(\pm\sqrt{113}, 0)$. At $P(8 \sec \theta, 7 \tan \theta)$, $\frac{dy}{dx} = \frac{7 \sec \theta}{8 \tan \theta}$ and the tangent is $\frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = \sec^2 \theta - \tan^2 \theta = 1$, meeting the y -axis at $Q = (0, -7 \cot \theta)$.

2 Practice

2.1 For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

2.2 For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find the eccentricity, the foci and the directrices. [3]

2.3 Find the equations of the tangents to $\frac{x^2}{4} + \frac{y^2}{3} = 1$ with gradient $\frac{1}{2}$. [4]

2.4 Show that the point $(4 \sec \theta, 3 \tan \theta)$ lies on $\frac{x^2}{16} - \frac{y^2}{9} = 1$ for all values of θ . [3]

3 Exam-style questions

3.1 (Edexcel IAL WFM03 June 2025 Q9a–c, condensed.) The hyperbola H has equation $\frac{x^2}{64} - \frac{y^2}{49} = 1$ and eccentricity e . (a) Show that $e = \frac{\sqrt{113}}{8}$. (b) Use calculus to show that the tangent to H at $P(8 \sec \theta, 7 \tan \theta)$ has equation $\frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = 1$. (c) Write down the coordinates of the point Q where this tangent meets the y -axis. [7]

3.2 (*Original.*) The hyperbola K has equation $\frac{x^2}{49} - \frac{y^2}{32} = 1$. (a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7 \sec \theta, 4\sqrt{2} \tan \theta)$ lies on K for all θ . [5]

Solutions

2.1 $9 = 25(1 - e^2)$ gives $e = \frac{4}{5}$. Foci $(\pm ae, 0) = (\pm 4, 0)$; directrices $x = \pm \frac{a}{e} = \pm \frac{25}{4}$.

2.2 $16 = 9(e^2 - 1)$ gives $e^2 = \frac{25}{9}$, $e = \frac{5}{3}$. Foci $(\pm 5, 0)$; directrices $x = \pm \frac{9}{5}$.

2.3 With $m = \frac{1}{2}$: $c^2 = a^2 m^2 + b^2 = 4 \cdot \frac{1}{4} + 3 = 4$, so $c = \pm 2$ and the tangents are $y = \frac{1}{2}x \pm 2$. (Substituting either into the ellipse gives a quadratic with zero discriminant.)

2.4 $\frac{(4 \sec \theta)^2}{16} - \frac{(3 \tan \theta)^2}{9} = \sec^2 \theta - \tan^2 \theta = 1$ for all θ ✓.

3.1 (a) $a^2 = 64$, $b^2 = 49$: $49 = 64(e^2 - 1)$, so $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$. (b) $x = 8 \sec \theta$, $y = 7 \tan \theta$: $\frac{dy}{dx} = \frac{7 \sec^2 \theta}{8 \sec \theta \tan \theta}$. The tangent: $y - 7 \tan \theta = \frac{7 \sec \theta}{8 \tan \theta}(x - 8 \sec \theta)$; multiplying through by $\frac{8 \tan \theta}{7}$ and collecting, $\frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = \sec^2 \theta - \tan^2 \theta = 1$. (c) At $x = 0$: $-\frac{y}{7} \tan \theta = 1$, so $Q = (0, -7 \cot \theta)$.

3.2 (a) $a^2 = 49$, $b^2 = 32$: $32 = 49(e^2 - 1)$, so $e^2 = \frac{81}{49}$, $e = \frac{9}{7}$; foci $(\pm ae, 0) = (\pm 9, 0)$. (b) $\frac{(7 \sec \theta)^2}{49} = \sec^2 \theta$ and $\frac{(4\sqrt{2} \tan \theta)^2}{32} = \frac{32 \tan^2 \theta}{32} = \tan^2 \theta$, so the left side is $\sec^2 \theta - \tan^2 \theta = 1$ ✓.

Local review guidance: correct b^2 - e relation for the curve type (sign!); foci and directrices placed with ae and a/e ; tangents from $\frac{dy}{dx}$ at the parametric point, or the $c^2 = a^2 m^2 \pm b^2$ condition for given gradients. This is authored classroom guidance.