

10.2 Minimum spanning trees and shortest paths

Name: _____ Class: _____ Date: _____

Total: 24 marks

KEY WORDS minimum spanning tree • Dijkstra's algorithm

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Objective

Find minimum spanning trees by Kruskal and Prim (node and matrix forms) and run Dijkstra with labels in order of permanence, recovering routes (objectives 2.1–2.2).

1 Key knowledge and worked examples

MST. Kruskal: sort all edges, add the smallest that does not form a cycle (reject others); Prim: from a start vertex repeatedly add the cheapest edge joining the tree to a new vertex; the matrix form crosses out rows and rings cheapest column entries. Dijkstra: permanent labels smallest-first; update neighbours (new = permanent + weight if smaller); recover the route by tracing back.

Worked example (original). Edges AB 4, AC 2, BC 1, BD 5, CD 8. Kruskal: BC 1 ✓, AC 2 ✓ (the tree holds A, B, C), AB 4 forms a cycle (rejected), BD 5 adds D ✓, CD 8 rejected (cycle): MST = BC + AC + BD, total 8. Prim from A: AC 2, CB 1, BD 5 – the same tree. Dijkstra A to D: A 0, C 2, B 3, D 8; route A–C–B–D.

2 Practice

2.1 Edges AB 3, AC 1, BC 2, CD 4, AD 5: find the MST by Kruskal, listing edges in order of choice, and give its total weight. [3]

2.2 For the same network, run Prim from vertex B and show the MST is the same. [3]

2.3 Edges AB 2, BC 3, CD 1, AD 4, AC 2: use Dijkstra from A to find the shortest distance to D, showing each label as it becomes permanent, and state the route. [4]

3 Exam-style questions

3.1 (*Original.*) A network connects four villages P, Q, R, S with edge weights PQ 7, PR 3, PS 8, QR 2, RS 6, QS 5 (km). (a) Use Kruskal's algorithm to find the minimum spanning tree, listing the edges in the order chosen and rejecting cycles explicitly. (b) Use Prim's algorithm from P to find the same tree. (c) State the total weight of the MST and explain why both algorithms must agree. [8]

3.2 (*Original.*) With the same network, use Dijkstra's algorithm from P to find the shortest route from P to S, showing the order in which labels become permanent and the final route with its length. [6]

Solutions

2.1 Sorted: AC 1, BC 2, AB 3, CD 4, AD 5. Take AC(1) ✓; BC(2) ✓ (B joins); AB(3) – cycle (A–C–B) reject; CD(4) ✓ (D joins); AD(5) – cycle reject. MST = AC + BC + CD, total 7.

2.2 Prim from B: cheapest edge from {B} is BC(2); from {B,C} cheapest to a new vertex is CA(1); from {B,C,A} cheapest to D is CD(4). Same tree, total 7 ✓.

2.3 A = 0 permanent. Neighbours: B 2, C 2. B permanent (2) – D has no edge from B, so no update. C permanent (2; the tie with B may be broken in either order); update D: $2 + 1 = 3$, smaller than the direct AD 4. D permanent (3): shortest A to D = 3, route A–C–D ($2 + 1 = 3 < 4$ ✓).

3.1 (a) Sorted: QR 2, PR 3, QS 5, RS 6, PQ 7, PS 8. Take QR(2) ✓; PR(3) ✓ (P joins); QS(5) ✓ (S joins); RS(6) – cycle reject; PQ(7) – cycle reject; PS(8) – cycle reject. MST = QR + PR + QS, total 10. (b) Prim from P: PR(3); from {P,R} cheapest new edge RQ(2); from {P,R,Q} cheapest to S is QS(5). Same tree, total 10 ✓. (c) Both construct a spanning tree of least total weight; any MST of a network has the same minimum total, so the totals agree (the edges can differ only when weights tie).

3.2 P = 0 permanent. Update: R 3, Q 7, S 8. R permanent (3); update Q: $3 + 2 = 5$ (smaller than 7), S: $3 + 6 = 9$ (8 stays). Q permanent (5); update S via Q: $5 + 5 = 10 > 8$, so S keeps 8. S permanent (8). Shortest P to S = 8, route P–S (the alternatives P–R–S = $3 + 6 = 9$ and P–R–Q–S = $3 + 2 + 5 = 10$ are both longer ✓).

Local review guidance: edges chosen in weight order with cycle rejections shown; Prim grows by cheapest new-vertex edge, not cheapest free edge; Dijkstra labels in permanence order with updates shown. This is authored classroom guidance.