

1.2 Complex roots of cubics and quartics

Name: _____ Class: _____ Date: _____ **Total: 28 marks**

KEY WORDS conjugate pair • quadratic factor • discriminant

Complex roots of cubics and quartics

Objective

Use the conjugate-root property to find all roots of cubic and quartic equations with real (integer) coefficients, building the real quadratic factor and dividing it out (FP1 1.5–1.6).

1 Key knowledge and worked examples

Conjugate pairs. If $f(z_1) = 0$ for a polynomial with real coefficients, then $f(\bar{z}_1) = 0$. From a root $p \pm qi$ build the factor $z^2 - 2pz + (p^2 + q^2)$ – sum $2p$, product $p^2 + q^2$ – then divide or compare coefficients to finish.

Worked example – a quartic with one complex root (WFM01 June 2025 Q7). $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3 + 5i$. The conjugate $3 - 5i$ is also a root, giving the factor $z^2 - 6z + 34$. Writing $f(z) = (z^2 - 6z + 34)(Pz^2 + bz + c)$: $34c = 68 \Rightarrow c = 2$; $-6c + 34b = 192 \Rightarrow b = 6$; $b - 6P = -36 \Rightarrow P = 7$; $c - 6b + 34P = Q \Rightarrow Q = 204$. The remaining quadratic $7z^2 + 6z + 2$ has discriminant $36 - 56 = -20$, so the other roots are $z = \frac{-3 \pm i\sqrt{5}}{7}$.

Worked example – a cubic from one real root. $g(x) = x^3 - 7x^2 + 17x - 15$ has $x = 3$ as a root. Divide: $g(x) = (x - 3)(x^2 - 4x + 5)$, and the quadratic has discriminant $16 - 20 < 0$, so the roots are $x = 3, 2 \pm i$. Check the product of roots: $3(2+i)(2-i) = 3 \cdot 5 = 15 \checkmark$ (constant -15 with leading coefficient 1 needs $\alpha\beta\gamma = 15$).

2 Practice

2.1 Given that $2 + i$ is a root of $x^2 + px + q = 0$ with p, q real, find p and q . [2]

2.2 $f(x) = x^3 - 3x^2 + 7x - 5$ has root $x = 1$. Find all roots of $f(x) = 0$. [4]

2.3 One root of $x^4 - 2x^3 + 6x^2 - 2x + 5 = 0$ is $1 + 2i$. Find the other three roots. [5]

2.4 A quartic with real coefficients has roots $1 \pm 3i$ and a repeated real root k , with constant term 50. Find k . [3]

3 Exam-style questions

3.1 (*Original.*) $h(z) = 2z^4 + 2z^3 + 7z^2 + 4z + 15$ has a root $\frac{1}{2} + \frac{3}{2}i$. (a) Write down the real quadratic factor coming from this root. (b) Given $h(z) = (2z^2 - 2z + 5)(z^2 + mz + n)$, find m and n . (c) Hence solve $h(z) = 0$ completely. [6]

3.2 (Adapted from Edexcel IAL WFM01 June 2025 Q7.) $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3 + 5i$. Without using a calculator, determine P , Q and the three other roots, giving them in simplest form. [8]

Solutions

2.1 The other root is $2 - i$: sum $4 = -p$, product $5 = q$. So $p = -4$, $q = 5$.

2.2 $f(1) = 1 - 3 + 7 - 5 = 0$ ✓. Divide: $x^3 - 3x^2 + 7x - 5 = (x - 1)(x^2 - 2x + 5)$.
 $x^2 - 2x + 5 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$. Roots: 1, $1 \pm 2i$. (Product: $1 \cdot 5 = 5$ ✓.)

2.3 $1 - 2i$ is also a root, giving the factor $x^2 - 2x + 5$. Dividing: $\frac{x^4 - 2x^3 + 6x^2 - 2x + 5}{x^2 - 2x + 5} = x^2 + 1$, since $(x^2 - 2x + 5)(x^2 + 1) = x^4 - 2x^3 + 6x^2 - 2x + 5$ ✓. The other roots are $\pm i$.

2.4 Product of roots $= (1 + 3i)(1 - 3i) \cdot k^2 = 10k^2$. For a monic quartic the product equals the constant term (sign + for degree 4): $10k^2 = 50 \Rightarrow k = \pm\sqrt{5}$.

3.1 (a) Sum 1, product $\frac{1}{4} + \frac{9}{4} = \frac{5}{2}$: factor $z^2 - z + \frac{5}{2}$, or $2z^2 - 2z + 5$ scaled. (b) Comparing $(2z^2 - 2z + 5)(z^2 + mz + n) = 2z^4 + 2z^3 + 7z^2 + 4z + 15$: constant $5n = 15 \Rightarrow n = 3$; z^3 : $2m - 2 = 2 \Rightarrow m = 2$. Check the rest: $(2z^2 - 2z + 5)(z^2 + 2z + 3) = 2z^4 + 2z^3 + 7z^2 + 4z + 15$ ✓. (c) $z^2 + 2z + 3 = 0$ gives $z = -1 \pm i\sqrt{2}$, so the roots are $\frac{1}{2} \pm \frac{3}{2}i$, $-1 \pm i\sqrt{2}$.

3.2 Factor $z^2 - 6z + 34$; comparing coefficients: $c = 2$, $b = 6$, $P = 7$, $Q = 204$; other roots from $7z^2 + 6z + 2 = 0$: $z = \frac{-6 \pm \sqrt{-20}}{14} = \frac{-3 \pm i\sqrt{5}}{7}$ (and the conjugate $3 - 5i$).

Local review guidance: conjugate stated and the quadratic factor formed from sum and product; coefficient comparison or division carried through exactly; complex roots simplified to $a + bi$ form. This is authored classroom guidance.