

Exercise walk-through

6.5

Vertical circular motion

Pearson Edexcel · International A-Level · Further Mathematics

3. Elastic strings

Elastic strings

$T = \frac{\lambda e}{L}$, energy = $\frac{\lambda e^2}{2L}$; a midpoint string is two half-strings with natural length halved.
 WME03 Q4: energy gain $\frac{1}{2}mga$ at $CD = \frac{3a}{2}$, and with resistance $\frac{1}{5}mg$ the launch
 $\sqrt{3ag/2}$ lands at $V = \sqrt{29ag/10}$. Q1's hinged rod: moments about A give $T = \frac{1}{2}mg$,
 then Hooke gives $L = \frac{24a}{25}$.

Check yourself

Check yourself

$a = f(x)$ via $v \, dv/dx$ with constants closed; Hooke's law and elastic energy for whole and half strings; hinged-rod moments with the tension; SHM proof, period, amplitude, slow-time fractions; two-resolution horizontal circles; vertical-circle energy, $R = 0$, projectile continuation; hemisphere COM derivation; signed-mass composites; tilted equilibrium.

Exercises 2.1·2.2

2.1 A particle on a string (length r) swings in a vertical circle. Write the condition for completing the circle when the speed at the top is just sufficient. [2]

2.2 A bead slides from rest at the top of a smooth sphere of radius a . Show it leaves the surface when $\cos \theta = \frac{2}{3}$ (θ from the upward vertical). [4]

Exercises 2.1·2.2

2.1 A particle on a string (length r) swings in a vertical circle. Write the condition for completing the circle when the speed at the top is just sufficient. [2]

2.1 At the top, gravity alone provides the centripetal force: $mg = \frac{mv_{top}^2}{r}$, i.e. $v_{top}^2 = gr$; the speed at the bottom must then be $\sqrt{5gr}$ by energy.

2.2 A bead slides from rest at the top of a smooth sphere of radius a . Show it leaves the surface when $\cos \theta = \frac{2}{3}$ (θ from the upward vertical). [4]

2.2 Energy from the top: $\frac{1}{2}mv^2 = mga(1 - \cos \theta)$. Leaving ($R = 0$): $mg \cos \theta = \frac{mv^2}{a}$, so $\cos \theta = 2(1 - \cos \theta)$ and $3 \cos \theta = 2$ ✓.

Exercise 2.3

2.3 A particle is projected at speed $\sqrt{4ga}$ from the lowest point of a smooth cylinder (radius a , outside surface). Find where it leaves. [3]

Exercise 2.3

2.3 A particle is projected at speed $\sqrt{4ga}$ from the lowest point of a smooth cylinder (radius a , outside surface). Find where it leaves. [3]

2.3 Measure θ from the upward vertical. The particle climbs from the bottom ($\theta = 180^\circ$) to height $a(1+\cos\theta)$, so energy from launch gives $v^2 = 4ga - 2ga(1+\cos\theta) = 2ga(1-\cos\theta)$. On the outside of the cylinder the inward direction points towards the axis; the weight's component along it is $mg\cos\theta$ and the reaction R points outward: $mg\cos\theta - R = \frac{mv^2}{a}$. The particle leaves when $R = 0$: $ga\cos\theta = 2ga(1-\cos\theta)$, so $\cos\theta = \frac{2}{3}$ – it leaves at $\theta \approx 48^\circ$ from the top, having climbed $\frac{5}{3}a$, still with $v^2 = \frac{2ga}{3} > 0$ ✓.

Exercise 3.1

3.1 (*Edexcel IAL WME03 June 2025 Q5.*) A fixed solid sphere has centre O and radius r . A particle P of mass m is held at rest at A on the smooth outer surface, where OA makes an angle α ($\alpha < 45^\circ$) with the upward vertical. P is released from rest and leaves the surface at B , where OB makes an angle α with the horizontal, moving with speed v . Air resistance is negligible. (a) Show that $v^2 = 2gr(\cos \alpha - \sin \alpha)$. (b) Show that $\tan \alpha = \frac{2}{3}$. (c) At the instant P crosses the horizontal plane through O it moves at angle θ to the horizontal. Show that $\cos \theta = \frac{2}{\sqrt{39}}$. [14]

Exercise 3.1

3.1 (*Edexcel IAL WME03 June 2025 Q5.*) A fixed solid sphere has centre O and radius r . A particle P of mass m is held at rest at A on the smooth outer surface, where OA makes an angle α ($\alpha < 45^\circ$) with the upward vertical. P is released from rest and leaves the surface at B, where OB makes an angle α with the horizontal, moving with speed v . Air resistance is negligible. (a) Show that $v^2 = 2gr(\cos \alpha - \sin \alpha)$. (b) Show that $\tan \alpha = \frac{2}{3}$. (c) At the instant P crosses the horizontal plane through O it moves at angle θ to the horizontal. Show that $\cos \theta = \frac{2}{\sqrt{39}}$. [14]

3.1 (a) The height of A above O is $r \cos \alpha$; the height of B above O is $r \sin \alpha$; P falls $r(\cos \alpha - \sin \alpha)$, so $\frac{1}{2}mv^2 = mgr(\cos \alpha - \sin \alpha)$ ✓. (b) At B the reaction is zero and the component of weight towards O is $mg \sin \alpha$ (OB at α to the horizontal): $mg \sin \alpha = \frac{mv^2}{r}$. Substituting (a): $\sin \alpha = 2(\cos \alpha - \sin \alpha)$, so $3 \sin \alpha = 2 \cos \alpha$ and $\tan \alpha = \frac{2}{3}$ ✓. (c) From (b), $\cos \alpha = \frac{3}{\sqrt{13}}$, $\sin \alpha = \frac{2}{\sqrt{13}}$, and $v^2 = \frac{2gr}{\sqrt{13}}$. After B the horizontal component $v \sin \alpha$ is conserved: $v^2 \sin^2 \alpha = \frac{8gr}{13\sqrt{13}}$. At the horizontal through O, energy from A: $\frac{1}{2}mW^2 = mgr \cos \alpha$, so $W^2 = \frac{6gr}{\sqrt{13}}$. Then $\cos^2 \theta = \frac{v^2 \sin^2 \alpha}{W^2} = \frac{8}{78} = \frac{4}{39}$, giving $\cos \theta = \frac{2}{\sqrt{39}}$ ✓.

Local review guidance: heights measured from the centre with the stated angles; the $R = 0$ substitution; (dM1) horizontal-component conservation after leaving. This is authored classroom guidance.