Exercise walk-through

3.5
Reduction formulae, arc length and surfaces of revolution

Pearson Edexcel · International A-Level · Further Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c$, $\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c$, $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$,
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + c$. Complete the square first: $4x^2 + 8x + 9 = 4((x+1)^2 + \frac{5}{4})$.
Inverse functions: parts with u the inverse, e.g.
$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2 - 1} + c$.

10. Further matrix algebra

Matrices and eigenvalues

3×3 $\det$ by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row by row: components of $\mathbf{Mv}$ equal λv_j. For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$: $\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2. Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T\mathbf{AP}$ diagonal.

Check yourself

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Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.2

2.1 Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$, giving an exact answer. [4]

2.2 The curve $y = \cosh x$, $0 \leq x \leq \ln 2$, is rotated through 2π radians about the x-axis. Find, exactly, the area of the curved surface generated. [4]

Exercises 2.1·2.2

2.1 Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$, giving an exact answer. [4]

2.1 $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$, so with $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}dx$: $s = \displaystyle\int_0^4 \sqrt{1 + \tfrac{9}{4}x}\,dx = \tfrac{4}{9}\left[\tfrac{2}{3}u^{3/2}\right]_1^{10} = \tfrac{8}{27}\left(10^{3/2} - 1\right)$.

2.2 The curve $y = \cosh x$, $0 \le x \le \ln 2$, is rotated through 2π radians about the x-axis. Find, exactly, the area of the curved surface generated. [4]

2.2 $\frac{dy}{dx} = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$, so

$$S = 2\pi \int_0^{\ln 2} \cosh^2 x\,dx = 2\pi\left[\tfrac{1}{2}\sinh x \cosh x + \tfrac{x}{2}\right]_0^{\ln 2}$$

(from $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$). With $\sinh(\ln 2) = \frac{3}{4}$, $\cosh(\ln 2) = \frac{5}{4}$: $S = 2\pi\left(\frac{15}{32} + \frac{1}{2}\ln 2\right)$, i.e. $\frac{15\pi}{16} + \pi \ln 2$.

Exercise 2.3

2.3 Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. (a) Show that, for $n \geq 2$, $nI_n = (n-1)I_{n-2}$. (b) Hence find I_4. [5]

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2.3 (a) Parts with $u = \sin^{n-1} x$, $dv = \sin x \, dx$: $I_n = (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x \, dx = (n-1)(I_{n-2} - I_n)$ (the boundary term vanishes at both limits), so $nI_n = (n-1)I_{n-2}$. (b) $I_0 = \frac{\pi}{2}$, $I_2 = \frac{1}{2}I_0 = \frac{\pi}{4}$, $I_4 = \frac{3}{4}I_2 = \frac{3\pi}{16}$.

Exercise 3.1

3.1 *(Edexcel IAL WFM03 June 2025 Q5.)* $I_n = \int_1^6 x^n(3x-2)^{-1/2}\,dx$, $n \geq 0$. (a) Show that, for $n \geq 1$, $(3+6n)I_n = 4nI_{n-1} + 8 \times 6^n - 2$. (b) Use the reduction formula in part (a) to determine the exact value of $\int_1^6 x^3(3x-2)^{-1/2}\,dx$. [9]

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3.1 (a) $v = \frac{2}{3}\sqrt{3x-2}$ (check by differentiating). Parts:

$$I_n = \left[\tfrac{2}{3}x^n\sqrt{3x-2}\right]_1^6 - \tfrac{2n}{3}\int_1^6 x^{n-1}\sqrt{3x-2}\,dx$$

The boundary term: $\frac{2}{3}\left(6^n \cdot 4 - 1\right) = \frac{8 \cdot 6^n - 2}{3}$. The integral: $\int_1^6 \frac{x^{n-1}(3x-2)}{\sqrt{3x-2}}dx = 3I_n - 2I_{n-1}$. So $I_n = \frac{8 \cdot 6^n - 2}{3} - \frac{2n}{3}(3I_n - 2I_{n-1})$, which rearranges to $(3+6n)I_n = 4nI_{n-1} + 8 \cdot 6^n - 2$. (b) $I_0 = \frac{2}{3}[\sqrt{3x-2}]_1^6 = \frac{2}{3}(4-1) = 2$; $n = 1$: $9I_1 = 4(2) + 48 - 2 = 54$, so $I_1 = 6$; $n = 2$: $15I_2 = 8(6) + 8(36) - 2 = 334$, so $I_2 = \frac{334}{15}$; $n = 3$: $21I_3 = 12 \cdot \frac{334}{15} + 8(216) - 2 = \frac{1336}{5} + 1726 = \frac{9966}{5}$, so $I_3 = \frac{9966}{105} = \frac{3322}{35}$.

Exercise 3.2

3.2 *(Edexcel IAL WFM03 June 2025 Q7a.)* The curve with equation $y = \cos 2x$, $0 \leq x \leq \frac{\pi}{4}$, is rotated through 2π radians about the x-axis. Show that the area of the curved surface generated is $S = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4\sin^2 2x}\, dx$. [3]

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3.2 *(Edexcel IAL WFM03 June 2025 Q7a.)* The curve with equation $y = \cos 2x$, $0 \leq x \leq \frac{\pi}{4}$, is rotated through 2π radians about the x-axis. Show that the area of the curved surface generated is $S = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4\sin^2 2x}\, dx$. [3]

3.2 $\frac{dy}{dx} = -2\sin 2x$, so $\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + 4\sin^2 2x}$ and $S = 2\pi \int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4\sin^2 2x}\, dx$ (no modulus needed: $\cos 2x \geq 0$ on $[0, \frac{\pi}{4}]$).

Local review guidance: parts orientation chosen so the leftover integral re-expresses in I_n, I_{n-1}; boundary values at both limits; surface integrals carry the y factor and the surd from $\frac{dy}{dx}$, with exact evaluations. This is authored classroom guidance.