Exercise walk-through

3.4 Integrating hyperbolic and inverse functions

Pearson Edexcel · International A-Level · Further Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c$, $\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c$, $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$,
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + c$. Complete the square first: $4x^2 + 8x + 9 = 4((x+1)^2 + \frac{5}{4})$.
Inverse functions: parts with u the inverse, e.g.
$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2 - 1} + c$.

10. Further matrix algebra

Matrices and eigenvalues

3×3 $\det$ by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row by row: components of $\mathbf{Mv}$ equal λv_j. For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$: $\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2. Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T\mathbf{AP}$ diagonal.

Check yourself

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Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.3

2.1 Find $\int \sinh^3 x \cosh x \, dx$. [3]

2.2 Evaluate $\int_0^1 \frac{dx}{x^2+4}$. [4]

2.3 Find $\int \frac{dx}{x^2-9}$. [4]

Exercises 2.1·2.3

2.1 Find $\int \sinh^3 x \cosh x \, dx$. [3]

2.1 With $u = \sinh x$: $\frac{1}{4}\sinh^4 x + c$.

2.2 Evaluate $\int_0^1 \frac{dx}{x^2+4}$. [4]

2.2 $\left[\frac{1}{2}\arctan\frac{x}{2}\right]_0^1 = \frac{1}{2}\arctan\frac{1}{2}$.

2.3 Find $\int \frac{dx}{x^2-9}$. [4]

2.3 $\frac{1}{2\cdot 3}\ln\left|\frac{x-3}{x+3}\right| + c = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + c$.

Exercises 2.4·3.1

2.4 Find $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$. [3]

3.1 *(Edexcel IAL WFM03 June 2025 Q2.)* Determine

(i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$

(ii) $\displaystyle\int \operatorname{arcosh} 3x \, dx$. [7]

Exercises 2.4·3.1

2.4 Find $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$. [3]

2.4 $\operatorname{arsinh}\dfrac{x}{3}+c\ \left(=\ln\left(x+\sqrt{x^2+9}\right)+c'\right)$.

3.1 *(Edexcel IAL WFM03 June 2025 Q2.)* Determine

(i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$

(ii) $\displaystyle\int \operatorname{arcosh} 3x\, dx$. [7]

3.1 (i) $4x^2+8x+9=(2x+2)^2+5$, so with $u=2x+2$, $dx=\frac{du}{2}$: $\frac{1}{2}\displaystyle\int\frac{du}{\sqrt{u^2+5}}=\frac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}}+c$. (ii) By parts, $u=\operatorname{arcosh}3x$, $dv=dx$: $x\operatorname{arcosh}3x-\displaystyle\int x\cdot\frac{3}{\sqrt{9x^2-1}}dx=x\operatorname{arcosh}3x-\frac{1}{3}\sqrt{9x^2-1}+c$ (because $\dfrac{d}{dx}\sqrt{9x^2-1}=\dfrac{9x}{\sqrt{9x^2-1}}$).

Exercise 3.2

3.2 *(Original.)* Evaluate $\int_0^1 \operatorname{arsinh} x \, dx$, giving your answer in exact form. [4]

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3.2 By parts: $[x \operatorname{arsinh} x]_0^1 - \displaystyle\int_0^1 \frac{x \, dx}{\sqrt{1+x^2}} = \ln(1+\sqrt{2}) - \left[\sqrt{1+x^2}\right]_0^1 = \ln(1+\sqrt{2}) - \sqrt{2} + 1.$

Local review guidance: square completed correctly under the surd (or linear substitution identified); by parts with the inverse function chosen as u; exact forms with the constant of integration on indefinite integrals. This is authored classroom guidance.