Exercise walk-through

3.2
The ellipse and the hyperbola

Pearson Edexcel · International A-Level · Further Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c$, $\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c$, $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$,
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + c$. Complete the square first: $4x^2 + 8x + 9 = 4((x+1)^2 + \frac{5}{4})$.
Inverse functions: parts with u the inverse, e.g.
$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2 - 1} + c$.

10. Further matrix algebra

Matrices and eigenvalues

3×3 $\det$ by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row by row: components of $\mathbf{Mv}$ equal λv_j. For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$: $\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2. Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T\mathbf{AP}$ diagonal.

Check yourself

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Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.3

2.1 For the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

2.2 For the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$, find the eccentricity, the foci and the directrices. [3]

2.3 Find the equations of the tangents to $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ with gradient $\frac{1}{2}$. [4]

Exercises 2.1·2.3

2.1 For the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

2.1 $9=25(1-e^2)$ gives $e=\frac{4}{5}$. Foci $(\pm ae,0)=(\pm 4,0)$; directrices $x=\pm\frac{a}{e}=\pm\frac{25}{4}$.

2.2 For the hyperbola $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$, find the eccentricity, the foci and the directrices. [3]

2.2 $16=9(e^2-1)$ gives $e^2=\frac{25}{9}$, $e=\frac{5}{3}$. Foci $(\pm 5,0)$; directrices $x=\pm\frac{9}{5}$.

2.3 Find the equations of the tangents to $\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$ with gradient $\frac{1}{2}$. [4]

2.3 With $m=\frac{1}{2}$: $c^2=a^2m^2+b^2=4\cdot\frac{1}{4}+3=4$, so $c=\pm 2$ and the tangents are $y=\frac{1}{2}x\pm 2$. (Substituting either into the ellipse gives a quadratic with zero discriminant.)

Exercises 2.4·3.1

2.4 Show that the point $(4\sec\theta, 3\tan\theta)$ lies on $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ for all values of θ. [3]

3.1 *(Edexcel IAL WFM03 June 2025 Q9a–c, condensed.)* The hyperbola H has equation $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1$ and eccentricity e. (a) Show that $e = \dfrac{\sqrt{113}}{8}$. (b) Use calculus to show that the tangent to H at $P(8\sec\theta, 7\tan\theta)$ has equation $\frac{x}{8}\sec\theta - \frac{y}{7}\tan\theta = 1$. (c) Write down the coordinates of the point Q where this tangent meets the y-axis. [7]

Exercises 2.4·3.1

2.4 Show that the point $(4\sec\theta, 3\tan\theta)$ lies on $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ for all values of θ. [3]

2.4 $\dfrac{(4\sec\theta)^2}{16} - \dfrac{(3\tan\theta)^2}{9} = \sec^2\theta - \tan^2\theta = 1$ for all θ ✓.

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3.1 (a) $a^2 = 64$, $b^2 = 49$: $49 = 64(e^2 - 1)$, so $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$. (b) $x = 8\sec\theta$, $y = 7\tan\theta$: $\dfrac{dy}{dx} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta}$. The tangent: $y - 7\tan\theta = \dfrac{7\sec\theta}{8\tan\theta}(x - 8\sec\theta)$; multiplying through by $\frac{8\tan\theta}{7}$ and collecting, $\frac{x}{8}\sec\theta - \frac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta = 1$. (c) At $x = 0$: $-\frac{y}{7}\tan\theta = 1$, so $Q = (0, -7\cot\theta)$.

Exercise 3.2

3.2 *(Original.)* The hyperbola K has equation $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$. (a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7\sec\theta, 4\sqrt{2}\tan\theta)$ lies on K for all θ. [5]

Exercise 3.2

3.2 *(Original.)* The hyperbola K has equation $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$. (a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7\sec\theta, 4\sqrt{2}\tan\theta)$ lies on K for all θ. [5]

3.2 (a) $a^2 = 49$, $b^2 = 32$: $32 = 49(e^2 - 1)$, so $e^2 = \frac{81}{49}$, $e = \frac{9}{7}$; foci $(\pm ae, 0) = (\pm 9, 0)$. (b) $\dfrac{(7\sec\theta)^2}{49} = \sec^2\theta$ and $\dfrac{(4\sqrt{2}\tan\theta)^2}{32} = \frac{32\tan^2\theta}{32} = \tan^2\theta$, so the left side is $\sec^2\theta - \tan^2\theta = 1$ ✓.

Local review guidance: correct b^2–e relation for the curve type (sign!); foci and directrices placed with ae and a/e; tangents from $\frac{dy}{dx}$ at the parametric point, or the $c^2 = a^2m^2 \pm b^2$ condition for given gradients. This is authored classroom guidance.