9.2 Estimators, bias and the sample mean

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS unbiased • standard error

Estimators, bias and the sample mean

Objective

Test whether a quantity is a statistic, compute expectation-based bias, solve paired unbiased-estimator equations, and use the distribution of the sample mean including recovering (μ, σ) from given probabilities (objectives 3.1–3.2).

1 Key knowledge and worked examples

Toolkit. Statistic: no unknown parameters. Bias $= \mathrm{E}(\hat{\theta}) - \theta$. $\bar{X} \sim N(\mu, \sigma^2/n)$: standard error $\sigma/\sqrt{n}$. Two probability statements about $\bar{X}$ invert to two z-equations in μ and $\sigma/\sqrt{n}$.

Worked example (WST03 June 2025 Q6, Q7). (i) $R = \frac{1}{2}X_1 + \frac{3}{4}X_{20}$ has $\mathrm{E}(R) = \frac{5}{4}\mu$ – biased by $\frac{\mu}{4}$. $S = \frac{aX_1 - bX_2}{2}$ and $T = \frac{a\sum_1^{14} X_i + b\sum_{15}^{20} X_i}{20}$ unbiased force $a - b = 2$ and $14a + 6b = 20$: $a = 1.6$, $b = -0.4$. (ii) $P(\bar{X} < 45.2) = 0.3446$ gives $\mu - 0.4(\sigma/\sqrt{n}) = 45.2$; $P(\bar{X} > 75.2) = 0.0179$ gives $\mu + 2.1(\sigma/\sqrt{n}) = 75.2$; subtracting, $\sigma/\sqrt{n} = 12$ (so $\sigma = 12\sqrt{n}$) and $\mu = 50$.

2 Practice

2.1 $T = \frac{1}{3}(X_1 + X_2 + X_3)$ from a population with mean μ: show T is an unbiased estimator of μ. [2]

2.2 For a population with variance σ^2, find $\mathrm{Var}(T)$ in 2.1 and comment on why three observations are better than one. [2]

2.3 $\bar{X}$ is the mean of 64 observations from $N(25, 100)$. Find $P(\bar{X} > 26.75)$. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WST03 June 2025 Q6.)* A random sample $X_1, X_2, \ldots, X_{20}$ is taken from a population with unknown mean μ and unknown variance σ^2. (a) Explain why $\frac{1}{2}X_1 + \frac{3}{4}X_{20}$ is a statistic. (b) Show that $R = \frac{1}{2}X_1 + \frac{3}{4}X_{20}$ is a biased estimator of μ. (c) Write down the bias in terms of μ. (d) Given that $S = \frac{aX_1 - bX_2}{2}$ and $T = \frac{a\sum_{i=1}^{14} X_i + b\sum_{i=15}^{20} X_i}{20}$ are each unbiased estimators of μ, where a and b are constants, find the values of a and b. [8]

3.2 *(Edexcel IAL WST03 June 2025 Q7.)* A random sample of n observations is taken from a normal distribution with mean μ and standard deviation σ; $\bar{X}_n$ denotes the sample mean. Given that $P(\bar{X}_n < 45.2) = 0.3446$ and $P(\bar{X}_n > 75.2) = 0.0179$, (a) show that $\sigma = 12\sqrt{n}$; (b) find the value of μ; (c) find $P(\bar{X}_n > 59)$. [11]

Solutions

2.1 $\mathrm{E}(T) = \frac{1}{3}(\mu + \mu + \mu) = \mu$ – unbiased ✓.

2.2 $\mathrm{Var}(T) = \frac{1}{9}(3\sigma^2) = \frac{\sigma^2}{3}$ – one third of a single observation's variance, so averaging three shrinks the standard deviation by a factor of $\sqrt{3}$.

2.3 $\bar{X} \sim N(25, \frac{100}{64})$: $z = \frac{1.75}{1.25} = 1.4$, so $P = 1 - 0.9192 = 0.0808$.

3.1 (a) Its value is computed solely from sample observations – it involves no unknown population parameters. (b) $\mathrm{E}(R) = \frac{1}{2}\mathrm{E}(X_1) + \frac{3}{4}\mathrm{E}(X_{20}) = (\frac{1}{2} + \frac{3}{4})\mu = \frac{5}{4}\mu \neq \mu$, so R is biased. (c) Bias $= \frac{5}{4}\mu - \mu = \frac{\mu}{4}$. (d) $\mathrm{E}(S) = \frac{a-b}{2}\mu = \mu$ gives $a - b = 2$; $\mathrm{E}(T) = \frac{14a+6b}{20}\mu = \mu$ gives $14a + 6b = 20$. Solving: $a = 1.6$, $b = -0.4$.

3.2 (a) Standardising: $P(\bar{X}_n < 45.2) = 0.3446$ means the z-value is -0.4, so $\mu - 0.4\frac{\sigma}{\sqrt{n}} = 45.2$; $P(\bar{X}_n > 75.2) = 0.0179$ means $z = 2.1$, so $\mu + 2.1\frac{\sigma}{\sqrt{n}} = 75.2$. Subtracting: $2.5\frac{\sigma}{\sqrt{n}} = 30$, so $\frac{\sigma}{\sqrt{n}} = 12$, i.e. $\sigma = 12\sqrt{n}$ ✓. (b) $\mu = 45.2 + 0.4(12) = 50$ (check: $75.2 - 2.1(12) = 50$ ✓). (c) SE $= 12$: $z = \frac{59-50}{12} = 0.75$, so $P = 1 - 0.7734 = 0.2266$.

Local review guidance: statistic defined by absence of parameters; bias in terms of the parameter; two z-equations from the table values before any algebra. This is authored classroom guidance.