

## 8.3 Piecewise densities and the uniform distribution

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Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 30 marks

**KEY WORDS** continuous uniform distribution • variance

### Piecewise densities and the uniform distribution

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#### Objective

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Normalise piecewise-linear densities by exact algebra (calculator-banned style), compute moments by integration, and use the uniform distribution's facts both directions (objectives 2.2, 2.4, 3.1).

#### 1 Key knowledge and worked examples

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**Two shapes.** A piecewise-linear pdf normalises to (usually) a perfect square in the join point  $k$  – factorise it. Uniform on  $[a, b]$ : density  $\frac{1}{b-a}$ , mean  $\frac{a+b}{2}$ , variance  $\frac{(b-a)^2}{12}$ ,  $E(X^2) = \text{Var} + \text{mean}^2$ ; probabilities are lengths over  $b-a$ ; an interval probability gives  $b-a$ , the mean gives  $a+b$ .

**Worked example – the two shapes (WST02 June 2025 Q5, Q6).** (i)  $f = \frac{2x}{21}$  on  $[0, k]$  then  $\frac{2(6-x)}{15}$  on  $(k, 6]$ : normalising gives  $4k^2 - 28k + 49 = 0$ , i.e.  $(2k-7)^2 = 0$ , so  $k = 3.5$ ; then  $E(X) = \frac{19}{6}$  and (with  $E(X^2) = \frac{277}{24}$ )  $\text{Var} = \frac{109}{72}$ . (ii) Uniform with  $P(2 < X < 5) = \frac{3}{16}$  and mean 6:  $b-a = 16$ ,  $a+b = 12 \rightarrow a = -2$ ,  $b = 14$ ; variance  $\frac{64}{3}$ ,  $E(X^2) = \frac{172}{3}$ ,  $P(\frac{3}{2}X - b > a) = P(X > 8) = \frac{3}{8}$ .

#### 2 Practice

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**2.1**  $X$  is uniform on  $[3, 11]$ . Write down the mean, the variance and  $P(5 < X < 9)$ . [3]

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**3.2** (*Edexcel IAL WST02 June 2025 Q6.*) The continuous random variable  $X$  is uniformly distributed on the interval  $[a, b]$ . Given that  $P(2 < X < 5) = \frac{3}{16}$  and  $E(X) = 6$ : (a) find the value of  $a$  and the value of  $b$ ; (b) find the value of the constant  $c$  for which  $E(cX + 1) = 3$ ; (c) write down the exact value of  $\text{Var}(X)$ ; (d) find the exact value of  $E(X^2)$ ; (e) find  $P(\frac{3}{2}X - b > a)$ . [10]

## Solutions

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**2.1** Mean 7; variance  $\frac{64}{12} = \frac{16}{3}$ ;  $P(5 < X < 9) = \frac{4}{8} = \frac{1}{2}$ .

**2.2**  $\frac{(b-a)^2}{12} = \frac{64}{3}$  gives  $b - a = 16$ ;  $a + b = 20$ ; so  $a = 2$ ,  $b = 18$ .

**2.3**  $\int_0^4 \frac{x}{c} = \frac{8}{c} = 1$  gives  $c = 8$ ;  $E(X) = \frac{1}{8} \int_0^4 x^2 = \frac{64}{24} = \frac{8}{3}$ .

**3.1** (a) Normalising:  $\int_0^k \frac{2x}{21} dx + \int_k^6 \frac{2(6-x)}{15} dx = 1$ , i.e.  $\frac{k^2}{21} + \frac{2}{15} \left[ 6x - \frac{x^2}{2} \right]_k^6 = \frac{k^2}{21} + \frac{2}{15} (18 - 6k + \frac{k^2}{2}) = 1$ . Multiplying by 105:  $5k^2 + 14(18 - 6k + \frac{k^2}{2}) = 105$ , so  $5k^2 + 252 - 84k + 7k^2 = 105$ :  $12k^2 - 84k + 147 = 0$ , i.e.  $4k^2 - 28k + 49 = 0 = (2k - 7)^2 = 0$ , so  $k = 3.5$  ✓. (b)  $E(X) = \frac{2}{21} \int_0^{3.5} x^2 dx + \frac{2}{15} \int_{3.5}^6 (6x - x^2) dx$ . The first:  $\frac{2}{21} \left[ \frac{x^3}{3} \right]_0^{3.5} = \frac{2}{21} \cdot \frac{343}{24} = \frac{49}{36}$  (exactly  $\frac{343}{252}$ ). The second:  $\frac{2}{15} \left[ 3x^2 - \frac{x^3}{3} \right]_{3.5}^6$  with  $3(3.5)^2 - \frac{(3.5)^3}{3} = \frac{147}{4} - \frac{343}{24} = \frac{539}{24}$ , so the bracket is  $36 - \frac{539}{24} = \frac{325}{24}$  and the term is  $\frac{650}{360} = \frac{65}{36}$ . Total:  $E(X) = \frac{49+65}{36} = \frac{114}{36} = \frac{19}{6}$  ✓ (awrt 3.17). (c)  $\text{Var} = \frac{277}{24} - \left(\frac{19}{6}\right)^2 = \frac{277}{24} - \frac{361}{36} = \frac{831-722}{72} = \frac{109}{72}$  (awrt 1.51).

**3.2** (a)  $\frac{3}{b-a} = \frac{3}{16}$  gives  $b - a = 16$ ;  $\frac{a+b}{2} = 6$  gives  $a + b = 12$ :  $2b = 28$ , so  $b = 14$ ,  $a = -2$  (accept  $a = 4.4$ ,  $b = 7.6$ ). (b)  $E(cX + 1) = 6c + 1 = 3$  gives  $c = \frac{1}{3}$ . (c)  $\text{Var}(X) = \frac{16^2}{12} = \frac{64}{3}$ . (d)  $E(X^2) = \frac{64}{3} + 36 = \frac{172}{3}$ . (e)  $\frac{3}{2}X - 14 > -2$  gives  $X > 8$ :  $P(X > 8) = \frac{14-8}{16} = \frac{3}{8}$ .

Local review guidance: normalisation with both integrals over their own limits and the exact factorisation; moments by integration before any subtraction; uniform answers as exact fractions with the chosen pair of roots used consistently. This is authored classroom guidance.