8.2 Continuous random variables

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS probability density function • mode

Continuous random variables

Objective

Normalise piecewise densities, move between f and F in both directions, find medians, quartiles and modes, compute E(X), E(X^2) and Var, and read skewness from mean–median–mode (objectives 2.1–2.5).

1 Key knowledge and worked examples

Calculus kit. $\int f = 1$ pins constants; $f = F'$; $F(x) = \int_{-\infty}^{x} f$. Median and quartiles solve $F = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$; the mode maximises f (calculus or corners). $E(X) = \int xf$, $E(X^2) = \int x^2 f$, $\text{Var} = E(X^2) - [E(X)]^2$. Skewness: mean < median < mode is negative skew; the reverse is positive.

Worked example – a quartic CDF (WST02 June 2025 Q1). $F(x) = k(6x^2 - x^4)$ on [0, 1]: $F(1) = 1$ gives $k = \frac{1}{5}$; $P(X > 0.6) = 1 - F(0.6) = 0.594$; the median solves $2m^4 - 12m^2 + 5 = 0$ with $m = 0.671$; $f = \frac{12x - 4x^3}{5}$ and $f' = 0$ give the mode 1; mean 0.64 < median < mode means negative skew.

2 Practice

2.1 $f(x) = kx^2$ on [0, 2]. Find k, $E(X)$ and $\text{Var}(X)$. [4]

2.2 For the same variable, find the median. [2]

2.3 $f(x) = \frac{2x}{9}$ on $[0, 3]$. Find $P(X > 1)$ and $E(X)$. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WST02 June 2025 Q1, condensed.)* The continuous random variable X has cumulative distribution function $F(x) = k(6x^2 - x^4)$ for $0 \le x \le 1$ (0 below, 1 above). Given also that $E(X) = 0.64$: (a) show that $k = \frac{1}{5}$; (b) find $P(X > 0.6)$; (c) show that the median m satisfies $2m^4 - 12m^2 + 5 = 0$ and hence find m; (d) find, using calculus, the mode of X; (e) state, with a reason, the skewness of X. [11]

3.2 *(Original.)* $f(x) = kx^2(1 - x)$ on $[0, 1]$. (a) Show that $k = 12$. (b) Find $E(X)$ and $\text{Var}(X)$. (c) Find the mode of X. [6]

Solutions

2.1 $\int_0^2 kx^2 dx = \frac{8k}{3} = 1$ gives $k = \frac{3}{8}$. $E(X) = \frac{3}{8}\int_0^2 x^3 dx = \frac{3}{8}(4) = \frac{3}{2}$; $E(X^2) = \frac{3}{8}\int_0^2 x^4 dx = \frac{3}{8} \cdot \frac{32}{5} = \frac{12}{5}$; $\mathrm{Var}(X) = \frac{12}{5} - \frac{9}{4} = \frac{3}{20}$.

2.2 $F(m) = \frac{3}{8} \cdot \frac{m^3}{3} = \frac{m^3}{8} = \frac{1}{2}$: $m^3 = 4$, so $m = 1.59$ (3 s.f.).

2.3 $F(x) = \frac{x^2}{9}$: $P(X > 1) = 1 - \frac{1}{9} = \frac{8}{9}$. $E(X) = \frac{2}{9}\int_0^3 x^2 dx = \frac{2}{9}(9) = 2$.

3.1 (a) $F(1) = k(6-1) = 5k = 1$, so $k = \frac{1}{5}$ ✓. (b) $F(0.6) = \frac{6(0.36)-0.6^4}{5} = \frac{2.0304}{5} = 0.40608$, so $P(X > 0.6) = 0.59392$ (awrt 0.594). (c) $F(m) = \frac{1}{2}$: $6m^2 - m^4 = \frac{5}{2}$, i.e. $2m^4 - 12m^2 + 5 = 0$ ✓. Solving, $m^2 = \frac{12-\sqrt{144-40}}{4} = 3 - \frac{\sqrt{26}}{2} = 0.4505$, so $m = 0.671$ (the root inside [0, 1]). (d) $f(x) = F'(x) = \frac{12x-4x^3}{5}$; $f'(x) = \frac{12-12x^2}{5} = 0$ gives $x = \pm 1$ – only $x = 1$ lies in the support (where $f(1) = \frac{8}{5} > 0$), so the mode is 1. (e) mean $0.64 <$ median $0.671 <$ mode 1: negative skew.

3.2 (a) $\int_0^1 x^2(1-x)dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, so $k = 12$ ✓. (b) $E(X) = 12\int_0^1 (x^3 - x^4)dx = 12(\frac{1}{4} - \frac{1}{5}) = \frac{3}{5}$; $E(X^2) = 12(\frac{1}{5} - \frac{1}{6}) = \frac{2}{5}$; $\mathrm{Var}(X) = \frac{2}{5} - \frac{9}{25} = \frac{1}{25}$. (c) $f(x) = 12(x^2 - x^3)$; $f'(x) = 12(2x - 3x^2) = 0$ gives $x = \frac{2}{3}$ (and $x = 0$, an endpoint): the mode is $\frac{2}{3}$.

Local review guidance: normalisation over the correct limits; quartic medians solved for the root INSIDE the support; modes rejected outside the support. This is authored classroom guidance.