6.6 Centres of mass by integration

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS centre of mass • hemisphere

Centres of mass by integration

Objective

Derive centres of mass of solids of revolution by the disc integral, build composites with signed masses, and close tilted-equilibrium problems (objectives 5.1, 5.2).

1 Key knowledge and worked examples

Method. For a uniform solid of revolution about the x-axis, $\bar{x} = \dfrac{\int_0^a xy^2\,dx}{\int_0^a y^2\,dx}$ (weight thin discs by volume $\pi y^2\,dx$). The hemisphere: $y^2 = a^2 - x^2$ gives $\int_0^a x(a^2-x^2)dx = \frac{a^4}{4}$ over $\int_0^a (a^2 - x^2)dx = \frac{2a^3}{3}$, so $\bar{x} = \frac{3a}{8}$ from the plane face. Composites: mass ratios of similar solids are radii cubed; removed pieces negative. In tilted equilibrium the COM sits vertically above the contact point.

Worked example (WME03 June 2025 Q6). S = hemisphere radius a minus a radius-$\frac{a}{2}$ hemisphere (plane faces coplanar, centres O and C with $OC = \frac{a}{2}$): masses $8 : 1 : 7$; along OC, $8 \cdot \frac{3a}{8} - 1 \cdot \frac{3a}{16} = 7\bar{y}$ gives $\bar{y} = \frac{45a}{112}$; moments about the in-plane axis through O perpendicular to OC give the off-axis offset: the small hemisphere's COM projects to $\frac{a}{2}$ from O along OC, so $-\frac{a}{2} = 7\bar{x}$ and $\bar{x} = \frac{a}{14}$. Resting on its curved surface: $\tan\theta = \frac{a/14}{45a/112} = \frac{8}{45}$.

2 Practice

2.1 Show that the COM of a uniform solid cone of height h and base radius R is $\frac{h}{4}$ from the base. [3]

2.2 A solid hemisphere of radius a has a hemisphere of radius $\frac{a}{2}$ removed with the same plane face and centre. Find the COM of the remainder from the plane face. [3]

2.3 A uniform solid cylinder (radius a, length $4a$) rests with its curved surface on horizontal ground. Explain why every position with the axis horizontal is one of neutral equilibrium, and state the height of the centre of mass above the ground. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WME03 June 2025 Q6.)* (a) Show, using algebraic integration, that the centre of mass of a uniform solid hemisphere H of radius a is $\frac{3a}{8}$ from O, the centre of its plane face (the volume $\frac{2}{3}\pi a^3$ may be assumed). Solid S is formed by removing from H a solid hemisphere of radius $\frac{1}{2}a$, the plane face of the smaller hemisphere having centre C and lying in the plane face of H, with $OC = \frac{1}{2}a$. (b) Show that the centre of mass of S is $\frac{45a}{112}$ from the line through O and C. (c) S rests in equilibrium with its curved surface in contact with a rough horizontal plane. The angle between CO and the horizontal is θ. Find the exact value of $\tan\theta$. [14]

Solutions

2.1 With $y = \frac{R}{x}h$: $\bar{x} = \frac{\int_0^h x \cdot \frac{R^2x^2}{h^2}\,dx}{\int_0^h \frac{R^2x^2}{h^2}\,dx} = \frac{h^4/4}{h^3/3} = \frac{3h}{4}$ from the vertex, i.e. $\frac{h}{4}$ from the base ✓.

2.2 Masses $8:1:7$: $\bar{x} = \frac{8\cdot\frac{3a}{8} - 1\cdot\frac{3a}{16}}{7} = \frac{3a - \frac{3a}{16}}{7} = \frac{45a}{16\cdot 7} = \frac{45a}{112}$ from the common plane face.

2.3 Rolling the cylinder about its axis keeps the centre of mass (on the axis) at a constant height, so every such position is neutral equilibrium. The axis is a distance a (the radius) above the ground, so the centre of mass is at height a.

3.1 (a) Slicing perpendicular to the x-axis (measured from O along the symmetry axis): $\bar{x} = \frac{\int_0^a x(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx}$. Numerator: $[\frac{a^2x^2}{2} - \frac{x^4}{4}]_0^a = \frac{a^4}{2} - \frac{a^4}{4} = \frac{a^4}{4}$. Denominator: $[a^2x - \frac{x^3}{3}]_0^a = a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$ (so the volume is $\frac{2}{3}\pi a^3$ ✓). Hence $\bar{x} = \frac{a^4/4}{2a^3/3} = \frac{3a}{8}$ ✓. (b) The small hemisphere has radius $\frac{a}{2}$, so its mass ratio is $\frac{(a/2)^3}{a^3} = \frac{1}{8}$ and its COM is $\frac{3}{8}\cdot\frac{a}{2} = \frac{3a}{16}$ from O along OC. With masses $8:1:7$: $8\cdot\frac{3a}{8} - 1\cdot\frac{3a}{16} = 7\bar{y}$, i.e. $3a - \frac{3a}{16} = \frac{45a}{16} = 7\bar{y}$, so $\bar{y} = \frac{45a}{112}$ ✓. (c) Take the moment axis in the plane face, through O, perpendicular to OC. The big hemisphere's COM sits $\frac{3a}{8}$ above O measured perpendicular to the face, directly above O, so its in-plane offset from the axis is 0. The small hemisphere's COM sits above C, whose in-plane offset along OC is $\frac{a}{2}$ (its height $\frac{3a}{16}$ above the face is parallel to the axis and contributes nothing). Moments about the axis: $8\cdot 0 - 1\cdot\frac{a}{2} = 7\bar{x}$, so the COM of S is $\frac{a}{14}$ off the OC-line on the side away from C. Resting on its curved surface, the COM lies vertically above the contact point, so $\tan\theta = \dfrac{a/14}{45a/112} = \dfrac{8}{45}$ ✓.

Local review guidance: the disc integral with the correct limits and y^2; composite moments with cubed mass ratios; (dM1) the tilted rest closing with the COM vertically above the contact. This is authored classroom guidance.