6.5 Vertical circular motion

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS vertical circle

Vertical circular motion

Objective

Combine energy with Newton's law towards the centre for vertical circles: speeds at levels, leaving conditions ($R = 0$, $T = 0$) and projectile continuation conserving the horizontal component (objective 4.4).

1 Key knowledge and worked examples

The narrative. Energy between levels gives v^2 at each point; Newton's law towards the centre relates v^2 to the forces; the leaving condition on a surface is $R = 0$ (on a string, $T = 0$). After leaving, the particle is a projectile whose horizontal velocity component is conserved.

Worked example (WME03 June 2025 Q5). P released at A on a smooth sphere (radius r), OA at α to the upward vertical; leaves at B with OB at α to the horizontal. (a) A is $r\cos\alpha$ above O and B is $r\sin\alpha$ above O, so $\frac{1}{2}mv^2 = mgr(\cos\alpha - \sin\alpha)$. (b) At B, $R = 0$: $mg\sin\alpha = \frac{mv^2}{r} \Longrightarrow \sin\alpha = 2(\cos\alpha - \sin\alpha) \Longrightarrow \tan\alpha = \frac{2}{3}$. (c) After B, conserve $v\sin\alpha$ horizontally; at the horizontal through O, $\frac{1}{2}mW^2 = mgr\cos\alpha$, and $\cos^2\theta = \frac{v^2\sin^2\alpha}{W^2} = \frac{4}{39}$.

2 Practice

2.1 A particle on a string (length r) swings in a vertical circle. Write the condition for completing the circle when the speed at the top is just sufficient. [2]

2.2 A bead slides from rest at the top of a smooth sphere of radius a. Show it leaves the surface when $\cos\theta = \frac{2}{3}$ (θ from the upward vertical). [4]

2.3 A particle is projected at speed $\sqrt{4ga}$ from the lowest point of a smooth cylinder (radius a, outside surface). Find where it leaves. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WME03 June 2025 Q5.)* A fixed solid sphere has centre O and radius r. A particle P of mass m is held at rest at A on the smooth outer surface, where OA makes an angle α ($\alpha < 45°$) with the upward vertical. P is released from rest and leaves the surface at B, where OB makes an angle α with the horizontal, moving with speed v. Air resistance is negligible. (a) Show that $v^2 = 2gr(\cos\alpha - \sin\alpha)$. (b) Show that $\tan\alpha = \frac{2}{3}$. (c) At the instant P crosses the horizontal plane through O it moves at angle θ to the horizontal. Show that $\cos\theta = \frac{2}{\sqrt{39}}$. [14]

Solutions

2.1 At the top, gravity alone provides the centripetal force: $mg = \frac{mv_{top}^2}{r}$, i.e. $v_{top}^2 = gr$; the speed at the bottom must then be $\sqrt{5gr}$ by energy.

2.2 Energy from the top: $\frac{1}{2}mv^2 = mga(1-\cos\theta)$. Leaving ($R = 0$): $mg\cos\theta = \frac{mv^2}{a}$, so $\cos\theta = 2(1-\cos\theta)$ and $3\cos\theta = 2$ ✓.

2.3 Measure θ from the upward vertical. The particle climbs from the bottom ($\theta = 180°$) to height $a(1+\cos\theta)$, so energy from launch gives $v^2 = 4ga - 2ga(1+\cos\theta) = 2ga(1-\cos\theta)$. On the outside of the cylinder the inward direction points towards the axis; the weight's component along it is $mg\cos\theta$ and the reaction R points outward: $mg\cos\theta - R = \frac{mv^2}{a}$. The particle leaves when $R = 0$: $ga\cos\theta = 2ga(1-\cos\theta)$, so $\cos\theta = \frac{2}{3}$ – it leaves at $\theta \approx 48°$ from the top, having climbed $\frac{5}{3}a$, still with $v^2 = \frac{2ga}{3} > 0$ ✓.

3.1 (a) The height of A above O is $r\cos\alpha$; the height of B above O is $r\sin\alpha$; P falls $r(\cos\alpha - \sin\alpha)$, so $\frac{1}{2}mv^2 = mgr(\cos\alpha - \sin\alpha)$ ✓. (b) At B the reaction is zero and the component of weight towards O is $mg\sin\alpha$ (OB at α to the horizontal): $mg\sin\alpha = \frac{mv^2}{r}$. Substituting (a): $\sin\alpha = 2(\cos\alpha - \sin\alpha)$, so $3\sin\alpha = 2\cos\alpha$ and $\tan\alpha = \frac{2}{3}$ ✓. (c) From (b), $\cos\alpha = \frac{3}{\sqrt{13}}$, $\sin\alpha = \frac{2}{\sqrt{13}}$, and $v^2 = \frac{2gr}{\sqrt{13}}$. After B the horizontal component $v\sin\alpha$ is conserved: $v^2\sin^2\alpha = \frac{8gr}{13\sqrt{13}}$. At the horizontal through O, energy from A: $\frac{1}{2}mW^2 = mgr\cos\alpha$, so $W^2 = \frac{6gr}{\sqrt{13}}$. Then $\cos^2\theta = \dfrac{v^2\sin^2\alpha}{W^2} = \dfrac{8}{78} = \dfrac{4}{39}$, giving $\cos\theta = \frac{2}{\sqrt{39}}$ ✓.

Local review guidance: heights measured from the centre with the stated angles; the $R = 0$ substitution; (dM1) horizontal-component conservation after leaving. This is authored classroom guidance.