

## 2.3 Euler's relation, de Moivre's theorem and complex roots

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

**Total: 25 marks**

**KEY WORDS** Euler's relation

• de Moivre's theorem

• modulus

• argument

### Euler's relation, de Moivre's theorem and complex roots

#### Objective

Use Euler's relation and de Moivre's theorem to expand multiple-angle identities and powers of trigonometric functions, prove the theorem for integer  $n$ , and find all  $n$ th roots of a complex number in exact polar form (FP2 3.1–3.2).

#### 1 Key knowledge and worked examples

**Core relations.**  $e^{i\theta} = \cos \theta + i \sin \theta$ , so  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$  and  $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ .

De Moivre:  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ , provable for positive integers  $n$  by induction (the inductive step multiplies by  $\cos \theta + i \sin \theta$  and expands with the addition formulae); negative integers follow by taking reciprocals.

**Worked example – powers to multiple angles (WFM02 June 2025 Q9).** The imaginary part of  $(\cos \theta + i \sin \theta)^5$  is  $5c^4s - 10c^2s^3 + s^5$ ; substituting  $c^2 = 1 - s^2$  gives  $\sin 5\theta = 16\sin^5 \theta - 20\sin^3 \theta + 5\sin \theta$ . Setting  $5\theta = \pi$  yields  $16s^4 - 20s^2 + 5 = 0$ , so  $s^2 = \frac{5 \pm \sqrt{5}}{8}$ :  $\sin \frac{\pi}{5} = \sqrt{\frac{5 - \sqrt{5}}{8}}$  (the smaller root, since  $\sin \frac{\pi}{5} < \sin \frac{3\pi}{5}$ ).

**Worked example – fifth roots (WFM02 June 2025 Q1).** For  $z^5 = -2\sqrt{2} - 2\sqrt{6}i$ : modulus  $\sqrt{8 + 24} = 4\sqrt{2}$ , argument  $-\frac{2\pi}{3}$  (third quadrant, tangent  $\sqrt{3}$ ). So  $z = \sqrt{2} e^{i(-\frac{2\pi}{15} + \frac{2\pi k}{5})}$ ,  $k = 0, 1, 2, 3, 4$  – five roots spaced  $\frac{2\pi}{5}$  apart on  $|z| = \sqrt{2}$ . Exact polar form only.

## 2 Practice

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**2.1** Use de Moivre's theorem to express  $\cos 3\theta$  as a polynomial in  $\cos \theta$ . [3]

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**2.2** Express  $\sin^3 \theta$  in the form  $a \sin \theta + b \sin 3\theta$ . [3]

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**2.3** Solve  $z^3 = 8i$ , giving the roots in the form  $re^{i\theta}$  with exact  $\theta$ . [4]

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## 3 Exam-style questions

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**3.1** (*Original.*) (a) Show that  $\cos^4 \theta = \frac{1}{8}(3 + 4 \cos 2\theta + \cos 4\theta)$ . (b) Hence evaluate  $\int_0^{\pi/2} \cos^4 \theta \, d\theta$ . [7]

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**3.2** (*Edexcel IAL WFM02 June 2025 Q1.*) (a) Express  $-2\sqrt{2} - 2\sqrt{6}i$  in the form  $r(\cos \theta + i \sin \theta)$ , where  $r > 0$  and  $-\pi < \theta \leq \pi$ . (b) Hence solve  $z^5 = -2\sqrt{2} - 2\sqrt{6}i$ , giving your answers in the form  $re^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ . [8]

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## Solutions

**2.1** Real part of  $(c + is)^3$ :  $c^3 - 3cs^2 = c^3 - 3c(1 - c^2) = 4\cos^3\theta - 3\cos\theta$ .

**2.2** From  $\sin 3\theta = 3s - 4s^3$  (imaginary part of  $(c + is)^3$ ):  $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ .

**2.3**  $8i = 8e^{i\pi/2}$ , so  $z = 2e^{i(\pi/6+2\pi k/3)}$ ,  $k = 0, 1, 2$ :  $2e^{i\pi/6}$ ,  $2e^{i5\pi/6}$ ,  $2e^{i3\pi/2}$ . (Check: the product of the roots is  $8e^{i(\pi/6+5\pi/6+3\pi/2)} = 8e^{i5\pi/2} = 8i$  ✓, matching  $z_1 z_2 z_3 = -(-8i)$ .)

**3.1** (a)  $\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}) = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$ . (Or: real part of  $(\cos\theta + i\sin\theta)^4$  is  $c^4 - 6c^2s^2 + s^4$ ; substitute  $c^2 = \frac{1+\cos 2\theta}{2}$ ,  $s^2 = \frac{1-\cos 2\theta}{2}$ .) (b)

$$\int_0^{\pi/2} \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta) d\theta = \frac{1}{8} \left[ 3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta \right]_0^{\pi/2} = \frac{1}{8} \cdot \frac{3\pi}{2} = \frac{3\pi}{16}.$$

**3.2** (a) Modulus  $\sqrt{(-2\sqrt{2})^2 + (-2\sqrt{6})^2} = \sqrt{32} = 4\sqrt{2}$ . Both components negative, and  $\tan\theta = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$  gives a reference angle  $\frac{\pi}{3}$ , so  $\theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$ :  $4\sqrt{2}(\cos \frac{2\pi}{3} \cdot (-1) \dots)$ , i.e.  $4\sqrt{2}(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}))$ . (b)  $z = (4\sqrt{2})^{1/5} e^{i(\frac{-2\pi/3+2\pi k}{5})} = \sqrt{2} e^{i(-\frac{2\pi}{15} + \frac{2\pi k}{5})}$ ,  $k = 0, 1, 2, 3, 4$ , i.e. arguments  $-\frac{2\pi}{15}, \frac{4\pi}{15}, \frac{10\pi}{15} = \frac{2\pi}{3}, \frac{16\pi}{15} - 2\pi = -\frac{14\pi}{15}, \frac{22\pi}{15} - 2\pi = -\frac{8\pi}{15}$  (all in  $-\pi < \theta \leq \pi$ ):  $z \in \{\sqrt{2}e^{-2\pi i/15}, \sqrt{2}e^{4\pi i/15}, \sqrt{2}e^{2\pi i/3}, \sqrt{2}e^{-14\pi i/15}, \sqrt{2}e^{-8\pi i/15}\}$ .

Local review guidance: modulus and argument found exactly with the correct quadrant; roots carry the  $\frac{2\pi k}{n}$  spread and land inside the stated argument range; power identities simplified with  $c^2 + s^2 = 1$  substitutions. This is authored classroom guidance.