2.2 The method of differences

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS method of differences • partial fractions • closed form

The method of differences

Objective

Sum finite series by splitting the summand (partial fractions or factorial identities) so that consecutive terms cancel, writing the $r = 1, 2, n-1, n$ lines that display the collapse, and close follow-ups by algebra (FP2 2.1).

1 Key knowledge and worked examples

The mechanism. If $u_r = v_r - v_{r+1}$, then $\sum_{r=1}^{n} u_r = v_1 - v_{n+1}$: each middle term appears once positive and once negative. Show the collapse explicitly – write the first two and last two terms and strike through what cancels. Partial fractions supply the split for rational summands; $(r+1)! - r! = r \cdot r!$ supplies it for factorials.

Worked example – telescoping to a closed form (WFM02 June 2025 Q5). $\dfrac{1}{(r+1)(r+2)} = \dfrac{1}{r+1} - \dfrac{1}{r+2}$, so $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$ (everything from $\frac{1}{3}$ to $\frac{1}{n+1}$ cancels). Check $n = 1$: both sides $\frac{1}{6}$. A follow-up like "smallest n with sum > 0.49" is then algebra: $\dfrac{n}{2(n+2)} > 0.49 \Rightarrow n > 98$, so $n = 99$.

2 Practice

2.1 Show that $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$. [2]

2.2 Show that $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$. [4]

2.3 Show that $\displaystyle\sum_{r=1}^{n} r \cdot r! = (n+1)! - 1$. [3]

3 Exam-style questions

3.1 *(Original.)* (a) Show that $\dfrac{1}{r(r+1)(r+2)} = \dfrac{1}{2}\left[\dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}\right]$. (b) Hence find, in its simplest form, $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [6]

3.2 *(Adapted from Edexcel IAL WFM02 June 2025 Q5.)* (a) Express $\dfrac{1}{(r+1)(r+2)}$ in partial fractions. (b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$. (c) Determine the smallest value of n for which the sum exceeds 0.49. [10]

Solutions

2.1 $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$. The sum telescopes to $1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$.

2.2 $\dfrac{1}{(2r-1)(2r+1)} = \dfrac{1}{2}\left[\dfrac{1}{2r-1} - \dfrac{1}{2r+1}\right]$ (check: $\frac{1}{2} \cdot \frac{(2r+1)-(2r-1)}{(2r-1)(2r+1)} = \frac{1}{(2r-1)(2r+1)}$ ✓). The sum is $\frac{1}{2}\left(1 - \dfrac{1}{2n+1}\right) = \frac{1}{2} \cdot \dfrac{2n}{2n+1} = \dfrac{n}{2n+1}$.

2.3 $r \cdot r! = (r+1-1)r! = (r+1)! - r!$. The sum telescopes to $(n+1)! - 1! = (n+1)! - 1$. (Check $n = 2$: $1 + 2 \cdot 2 = 5 = 6 - 1$ ✓.)

3.1 (a) $\dfrac{1}{2}\left[\dfrac{(r+2)-r}{r(r+1)(r+2)}\right] = \dfrac{1}{r(r+1)(r+2)}$ ✓. (b) The sum is $\dfrac{1}{2}\left[\dfrac{1}{1 \cdot 2} - \dfrac{1}{(n+1)(n+2)}\right] = \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$. (Check $n = 1$: $\frac{1}{6}$ and $\frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ ✓.)

3.2 (a) $\dfrac{1}{(r+1)(r+2)} = \dfrac{1}{r+1} - \dfrac{1}{r+2}$. (b) Writing out $r = 1$ to n: $\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$. (c) $\dfrac{n}{2(n+2)} > 0.49 \Rightarrow n > 0.98(n+2) \Rightarrow 0.02n > 1.96 \Rightarrow n > 98$, so the smallest is $n = 99$ (check: $\frac{99}{202} \approx 0.4901 > 0.49$ ✓; $\frac{98}{200} = 0.49$ not strictly greater ✓).

Local review guidance: correct partial-fraction or factorial split verified; the cancellation displayed through first/last terms; closed form simplified and spot-checked. This is authored classroom guidance.