10.5 Linear programming

Name: _____ Class: _____ Date: _____ **Total: 27 marks**

KEY WORDS feasible region • linear programming

Linear programming

Objective

Formulate problems as linear programs (variables, objective, all constraints), solve two-variable problems graphically by ruler and vertex methods, and find best integer solutions (objectives 5.1–5.3).

1 Key knowledge and worked examples

Form then solve. Define x and y with units; write "maximise/minimise" for the objective; list EVERY constraint including $x, y \geq 0$. Graph the feasible region, then either slide a ruler along the objective direction (optimum at the last vertex touched) or evaluate the objective at each vertex. Integer answers: test integer points near the optimal vertex that lie inside the region.

Worked example (original). Maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $2x + y \leq 14$, $x, y \geq 0$. Vertices $(0, 0)$, $(7, 0)$, $(4, 6)$, $(0, 10)$ give $P = 0, 21, 24, 20$: the maximum is $P = 24$ at $(4, 6)$, both constraints tight.

2 Practice

2.1 A baker makes x plain and y iced buns. Flour: at most 200 kg, with plain buns using 2 kg per dozen and iced 4 kg per dozen. Time: at most 60 hours, with 1 hour per dozen plain and 1.5 hours per dozen iced. Write the linear program for maximum profit at 3 pounds per dozen plain and 5 pounds per dozen iced. [4]

2.2 For the constraints $x + y \leq 8$, $y \geq 2$, $x \leq 5$, $x, y \geq 0$, list the vertices of the feasible region. [3]

2.3 Minimise $C = 4x + 3y$ over the region of 2.2, using the vertex method. [3]

3 Exam-style questions

3.1 *(Original.)* A factory makes chairs (x) and tables (y). Each chair needs 2 hours of carpentry and each table 4 hours; at most 48 hours are available. Each chair needs 1 hour of finishing and each table 2 hours; at most 20 hours are available. Demand limits tables to at most 6. The profit is 40 pounds per chair and 70 pounds per table. (a) Formulate the linear program. (b) Solve graphically by the vertex method, listing every vertex and the profit there. (c) If chairs and tables must be whole units, confirm the optimal integer solution. [10]

3.2 *(Original.)* Minimise the cost $C = 5x + 4y$ subject to $x + 2y \geq 8$, $2x + y \geq 10$, $x, y \geq 0$. (a) Sketch the feasible region (it is unbounded towards large x, y) and list its vertices. (b) Find the minimum by the vertex method. (c) Explain why an unbounded region does not prevent a minimum here. [7]

Solutions

2.1 Maximise $P = 3x+5y$ subject to $2x+4y \leq 200$ (flour), $x+1.5y \leq 60$ (time), $x, y \geq 0$.

2.2 The region is bounded by $y = 2$ below, $x = 5$ right and $x + y = 8$ above: vertices $(0, 2)$ (y-axis meets $y = 2$), $(0, 8)$ (y-axis meets $x + y = 8$), $(5, 3)$ ($x = 5$ meets $x + y = 8$) and $(5, 2)$ ($x = 5$ meets $y = 2$).

2.3 C at the vertices: $(0, 2)$: 6; $(0, 8)$: 24; $(5, 3)$: 29; $(5, 2)$: 26. Minimum $C = 6$ at $(0, 2)$.

3.1 (a) Maximise $P = 40x+70y$ subject to $2x+4y \leq 48$, $x+2y \leq 20$, $y \leq 6$, $x, y \geq 0$. (b) Vertices and profits: $(0, 0)$: 0; $(10, 0)$: 400; $(8, 6)$ ($x+2y = 20$, $y = 6$): $40(8)+70(6) = 740$; $(6, 6)$ ($2x + 4y = 48$, $y = 6$): $40(6) + 70(6) = 660$; $(0, 6)$: 420. Maximum $P = 740$ at $(8, 6)$. (c) $(8, 6)$ is already integral: 8 chairs and 6 tables ✓ (check neighbouring integers $(8, 5)$: $P = 320 + 350 = 670 < 740$; $(9, 5)$: violates $x + 2y \leq 20$? $9 + 10 = 19 \leq 20$ ✓ and $2(9) + 4(5) = 38 \leq 48$ ✓, $P = 360 + 350 = 710 < 740$ ✓).

3.2 (a) The region lies above both lines; vertices: $(0, 10)$ (on $2x + y = 10$), $(4, 2)$ (intersection of $x + 2y = 8$ and $2x + y = 10$: solving, $y = 2$, $x = 4$), and $(8, 0)$ (on $x + 2y = 8$); the region extends to infinity up-right. (b) C at the vertices: $(0, 10)$: 40; $(4, 2)$: 28; $(8, 0)$: 40. Minimum $C = 28$ at $(4, 2)$. (c) Increasing x or y only adds cost (coefficients positive), so the unbounded directions cannot lower C; the minimum must occur at a vertex.

Local review guidance: every constraint captured with non-negativity; vertices found as constraint intersections, tested in the objective; integer checks at neighbouring lattice points inside the region. This is authored classroom guidance.