

## 1.3 Roots of quadratic equations

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 29 marks

**KEY WORDS** sum of roots • product of roots • symmetric expression

### Roots of quadratic equations

#### Objective

Use  $\alpha + \beta = -\frac{b}{a}$  and  $\alpha\beta = \frac{c}{a}$  to evaluate symmetric expressions and form new quadratic equations with transformed roots (FP1 2.1–2.3).

#### 1 Key knowledge and worked examples

**Symmetric toolkit.**  $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ ;  $\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ ;  $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$ . Never solve for the individual roots.

**Worked example – a new equation without solving (WFM01 June 2025 Q6).**  $f(x) = 3x^2 + kx - 5$  has roots  $\alpha, \beta$ :  $\alpha\beta = -\frac{5}{3}$ . The condition  $\alpha + \beta = 9\alpha\beta$  gives  $-\frac{k}{3} = -15$ , so  $k = 45$  and  $\alpha + \beta = -15$ . For new roots  $\alpha^2 + \beta^2$  and  $\alpha + \beta^2$ :

- sum  $= \alpha^2 + \beta^2 + \alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta + (\alpha + \beta) = 225 + \frac{10}{3} - 15 = \frac{640}{3}$
- product  $= (\alpha^3 + \beta^3) + \alpha\beta + (\alpha\beta)^2 = -3450 - \frac{5}{3} + \frac{25}{9} = -\frac{31040}{9}$

New equation:  $9x^2 - 1920x - 31040 = 0$ .

**Worked example – reciprocals.** Roots  $\frac{1}{\alpha}, \frac{1}{\beta}$  have sum  $\frac{\alpha + \beta}{\alpha\beta}$  and product  $\frac{1}{\alpha\beta}$ : for  $x^2 - 7x + 10$  (sum 7, product 10) the reciprocal-roots equation is  $x^2 - \frac{7}{10}x + \frac{1}{10} = 0$ , i.e.  $10x^2 - 7x + 1 = 0$ .

#### 2 Practice

**2.1** For  $2x^2 - 8x + 3 = 0$  with roots  $\alpha, \beta$ , find (a)  $\alpha + \beta$ ; (b)  $\alpha\beta$ ; (c)  $\alpha^2 + \beta^2$ ; (d)  $\frac{1}{\alpha} + \frac{1}{\beta}$ . [4]

**2.2** The equation  $x^2 + 6x + 4 = 0$  has roots  $\alpha, \beta$ . Without solving it, form a quadratic equation with roots  $3\alpha$  and  $3\beta$ . [3]

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**2.3** For  $x^2 - 5x + 9 = 0$  with roots  $\alpha, \beta$ , find the value of  $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ . [3]

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**2.4** (*Original.*)  $x^2 + px + q = 0$  has roots  $\alpha$  and  $\alpha + 2$ . Show that  $p^2 - 4q = 4$  for every such pair. [3]

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### 3 Exam-style questions

**3.1** (*Original.*)  $g(x) = 2x^2 - 6x + m = 0$  has roots  $\alpha, \beta$  with  $\alpha^2 + \beta^2 = 11$ . (a) Find  $m$ . (b) Form a quadratic equation with integer coefficients whose roots are  $\alpha^3$  and  $\beta^3$ . [6]

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**3.2** (*Adapted from Edexcel IAL WFM01 June 2025 Q6.*)  $f(x) = 3x^2 + kx - 5$  has roots  $\alpha, \beta$  with  $\alpha + \beta = 9\alpha\beta$ . (a) State  $\alpha\beta$  and find  $k$ . (b) Prove that  $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$  by expanding  $(\alpha + \beta)^3$ . (c) Without solving  $f(x) = 0$ , find a quadratic equation with integer coefficients whose roots are  $\alpha^2 + \beta$  and  $\alpha + \beta^2$ . [10]

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## Solutions

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**2.1** (a)  $\alpha + \beta = 4$ ; (b)  $\alpha\beta = \frac{3}{2}$ ; (c)  $16 - 3 = 13$ ; (d)  $\frac{4}{3/2} = \frac{8}{3}$ .

**2.2** New sum  $= 3(\alpha + \beta) = -18$ ; new product  $= 9\alpha\beta = 36$ . Equation:  $x^2 + 18x + 36 = 0$ .

**2.3**  $\frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{25 - 18}{9} = \frac{7}{9}$ .

**2.4** Sum  $2\alpha + 2 = -p$  and product  $\alpha(\alpha + 2) = q$ . Then  $p^2 = 4(\alpha + 1)^2 = 4\alpha^2 + 8\alpha + 4$  and  $4q = 4\alpha^2 + 8\alpha$ , so  $p^2 - 4q = 4$  ✓ – independent of  $\alpha$ .

**3.1** (a)  $(\alpha + \beta)^2 - 2\alpha\beta = 9 - m = 11 \Rightarrow m = -2$ . (b)  $\alpha\beta = -1$ ;  $\alpha^3 + \beta^3 = 27 - 3(-1)(3) = 36$ ;  $\alpha^3\beta^3 = (-1)^3 = -1$ . Equation:  $x^2 - 36x - 1 = 0$ .

**3.2** (a)  $\alpha\beta = -\frac{5}{3}$ ;  $-\frac{k}{3} = 9(-\frac{5}{3}) = -15 \Rightarrow k = 45$ ,  $\alpha + \beta = -15$ . (b)  $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = (\alpha^3 + \beta^3) + 3\alpha\beta(\alpha + \beta)$ ; rearrange. (c) Sum  $= \frac{640}{3}$ , product  $= -\frac{31040}{9}$  (see worked example):  $9x^2 - 1920x - 31040 = 0$ .

Local review guidance: symmetric identities applied without finding individual roots; exact fractions kept until the final integer equation; new equations presented as  $x^2 - (\text{sum})x + \text{product} = 0$ . This is authored classroom guidance.