1.2 Complex roots of cubics and quartics

Name: _____ Class: _____ Date: _____ **Total: 28 marks**

KEY WORDS conjugate pair • quadratic factor • discriminant

Complex roots of cubics and quartics

Objective

Use the conjugate-root property to find all roots of cubic and quartic equations with real (integer) coefficients, building the real quadratic factor and dividing it out (FP1 1.5–1.6).

1 Key knowledge and worked examples

Conjugate pairs. If $f(z_1) = 0$ for a polynomial with real coefficients, then $f(\bar{z}_1) = 0$. From a root $p \pm qi$ build the factor $z^2 - 2pz + (p^2 + q^2)$ – sum $2p$, product $p^2 + q^2$ – then divide or compare coefficients to finish.

Worked example – a quartic with one complex root (WFM01 June 2025 Q7). $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3 + 5i$. The conjugate $3 - 5i$ is also a root, giving the factor $z^2 - 6z + 34$. Writing $f(z) = (z^2 - 6z + 34)(Pz^2 + bz + c)$: $34c = 68 \Rightarrow c = 2$; $-6c + 34b = 192 \Rightarrow b = 6$; $b - 6P = -36 \Rightarrow P = 7$; $c - 6b + 34P = Q \Rightarrow Q = 204$. The remaining quadratic $7z^2 + 6z + 2$ has discriminant $36 - 56 = -20$, so the other roots are $z = \dfrac{-3 \pm i\sqrt{5}}{7}$.

Worked example – a cubic from one real root. $g(x) = x^3 - 7x^2 + 17x - 15$ has $x = 3$ as a root. Divide: $g(x) = (x - 3)(x^2 - 4x + 5)$, and the quadratic has discriminant $16 - 20 < 0$, so the roots are $x = 3,\ 2 \pm i$. Check the product of roots: $3(2+i)(2-i) = 3 \cdot 5 = 15$ ✓ (constant -15 with leading coefficient 1 needs $\alpha\beta\gamma = 15$).

2 Practice

2.1 Given that $2 + i$ is a root of $x^2 + px + q = 0$ with p, q real, find p and q. [2]

2.2 $f(x) = x^3 - 3x^2 + 7x - 5$ has root $x = 1$. Find all roots of $f(x) = 0$. [4]

2.3 One root of $x^4 - 2x^3 + 6x^2 - 2x + 5 = 0$ is $1 + 2i$. Find the other three roots. [5]

2.4 A quartic with real coefficients has roots $1 \pm 3i$ and a repeated real root k, with constant term 50. Find k. [3]

3 Exam-style questions

3.1 *(Original.)* $h(z) = 2z^4 + 2z^3 + 7z^2 + 4z + 15$ has a root $\frac{1}{2} + \frac{3}{2}i$. (a) Write down the real quadratic factor coming from this root. (b) Given $h(z) = (2z^2 - 2z + 5)(z^2 + mz + n)$, find m and n. (c) Hence solve $h(z) = 0$ completely. [6]

3.2 *(Adapted from Edexcel IAL WFM01 June 2025 Q7.)* $f(z) = Pz^4 - 36z^3 + Qz^2 + 192z + 68$ has root $3 + 5i$. Without using a calculator, determine P, Q and the three other roots, giving them in simplest form. [8]

Solutions

2.1 The other root is $2-i$: sum $4=-p$, product $5=q$. So $p=-4$, $q=5$.

2.2 $f(1)=1-3+7-5=0$ ✓. Divide: $x^3-3x^2+7x-5=(x-1)(x^2-2x+5)$. $x^2-2x+5=0 \Rightarrow x=\dfrac{2\pm\sqrt{4-20}}{2}=1\pm 2i$. Roots: $1,\ 1\pm 2i$. (Product: $1\cdot 5=5$ ✓.)

2.3 $1-2i$ is also a root, giving the factor x^2-2x+5. Dividing: $\dfrac{x^4-2x^3+6x^2-2x+5}{x^2-2x+5}=x^2+1$, since $(x^2-2x+5)(x^2+1)=x^4-2x^3+6x^2-2x+5$ ✓. The other roots are $\pm i$.

2.4 Product of roots $=(1+3i)(1-3i)\cdot k^2=10k^2$. For a monic quartic the product equals the constant term (sign + for degree 4): $10k^2=50 \Rightarrow k=\pm\sqrt{5}$.

3.1 (a) Sum 1, product $\frac{1}{4}+\frac{9}{4}=\frac{5}{2}$: factor $z^2-z+\frac{5}{2}$, or $2z^2-2z+5$ scaled. (b) Comparing $(2z^2-2z+5)(z^2+mz+n)=2z^4+2z^3+7z^2+4z+15$: constant $5n=15 \Rightarrow n=3$; z^3: $2m-2=2 \Rightarrow m=2$. Check the rest: $(2z^2-2z+5)(z^2+2z+3)=2z^4+2z^3+7z^2+4z+15$ ✓. (c) $z^2+2z+3=0$ gives $z=-1\pm i\sqrt{2}$, so the roots are $\frac{1}{2}\pm\frac{3}{2}i,\ -1\pm i\sqrt{2}$.

3.2 Factor $z^2-6z+34$; comparing coefficients: $c=2$, $b=6$, $P=7$, $Q=204$; other roots from $7z^2+6z+2=0$: $z=\dfrac{-6\pm\sqrt{-20}}{14}=\dfrac{-3\pm i\sqrt{5}}{7}$ (and the conjugate $3-5i$).

Local review guidance: conjugate stated and the quadratic factor formed from sum and product; coefficient comparison or division carried through exactly; complex roots simplified to $a+bi$ form. This is authored classroom guidance.