Exercise walk-through

2.5
First and second order differential equations

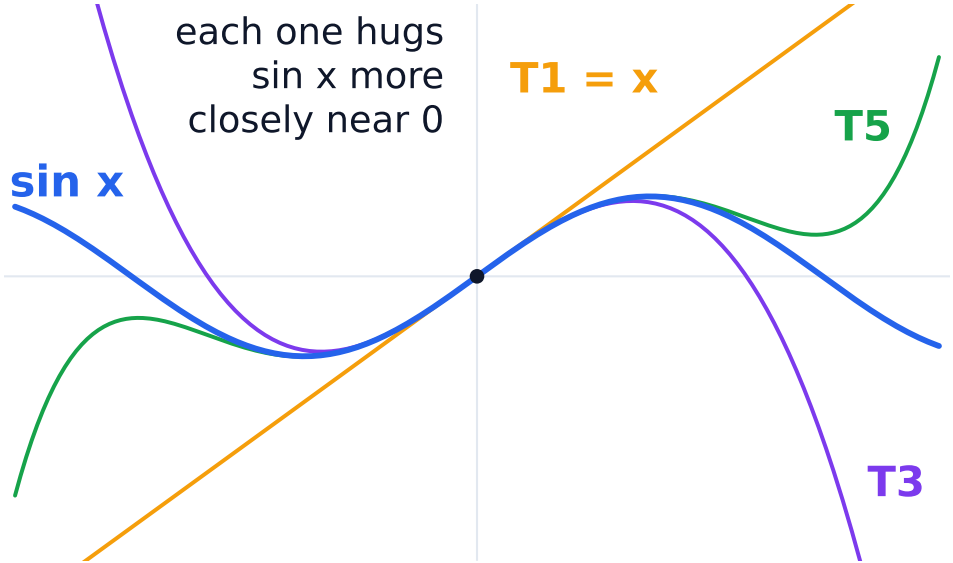
Pearson Edexcel · International A-Level · Further Mathematics

First order differential equations

- A first order linear differential equation has the form
- where P and Q are functions of x.
- Multiply every term by the **integrating factor** $e^{\int P\,dx}$.
- The left side then becomes the derivative of a product, $\frac{d}{dx}\left(y\,e^{\int P\,dx}\right)$, so you can integrate both sides.

$$\frac{dy}{dx} + Py = Q$$

Figure: Maclaurin and Taylor series



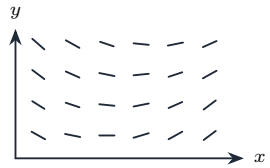
- The curves compare $\sin x$ with $T_1 = x$, $T_3 = x - x^3/6$ and $T_5 = x - x^3/6 + x^5/120$.
- Read the axes and units first.

Key terms (2)

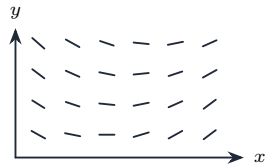
- **Integrating factor**: $e^{\int P\,dx}$, which turns the left side of $y' + Py = Q$ into the derivative of a product.
- **Complementary function**: the general solution of the equation with the right side set to zero.
- **Particular integral**: one solution of the complete differential equation.
- **Polar coordinates**: a distance from the pole and an angle from the **initial line**.

Exercises 2.1·2.2

2.1 Solve $\dfrac{dy}{dx} + 2y = e^{-x}$. [4]

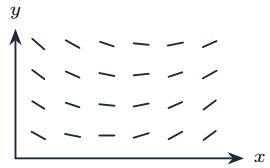


2.2 Solve $x\dfrac{dy}{dx} + y = x^2$ for $x > 0$. [3]



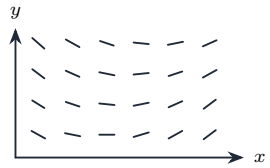
Exercises 2.1·2.2

2.1 Solve $\dfrac{dy}{dx} + 2y = e^{-x}$. [4]



2.1 IF $= e^{2x}$: $\dfrac{d}{dx}(ye^{2x}) = e^{2x}e^{-x} = e^{x}$, so $ye^{2x} = e^{x} + C$ and $y = e^{-x} + Ce^{-2x}$.

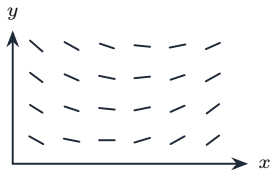
2.2 Solve $x\dfrac{dy}{dx} + y = x^2$ for $x > 0$. [3]



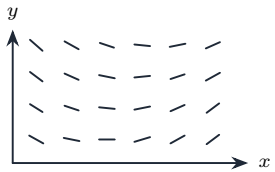
2.2 The left side is $\dfrac{d}{dx}(xy)$: $xy = \frac{x^3}{3} + C$, so $y = \frac{x^2}{3} + \dfrac{C}{x}$.

Exercises 2.3·2.4

2.3 Solve $y'' + y = 0$ given $y = 1$ and $y' = 0$ at $x = 0$. [2]

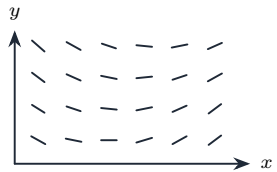


2.4 Find the general solution of $y'' - 4y = 12x$. [4]



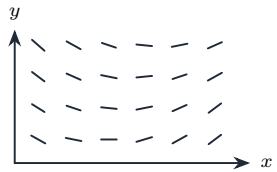
Exercises 2.3·2.4

2.3 Solve $y'' + y = 0$ given $y = 1$ and $y' = 0$ at $x = 0$. [2]



2.3 Auxiliary $m^2 + 1 = 0$: $y = A\cos x + B\sin x$. $y(0) = A = 1$; $y' = -A\sin x + B\cos x$, $y'(0) = B = 0$. So $y = \cos x$.

2.4 Find the general solution of $y'' - 4y = 12x$. [4]

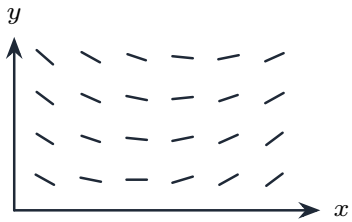


2.4 Auxiliary $m^2 - 4 = 0$: $m = \pm 2$, CF $= Ae^{2x} + Be^{-2x}$. Trial PI $= ax + b$: substitution gives $-4(ax + b) = 12x$, so $a = -3$, $b = 0$. General solution: $y = Ae^{2x} + Be^{-2x} - 3x$.

exercises.lol exercises.com.cn

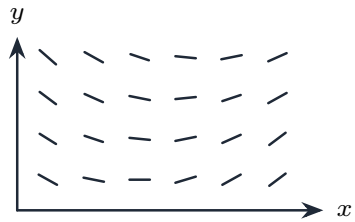
Exercise 3.1

3.1 *(Original.)* Find the general solution of $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-2x}$. [6]



Exercise 3.1

3.1 *(Original.)* Find the general solution of $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-2x}$. [6]



3.1 Auxiliary $m^2 + 4m + 4 = (m+2)^2 = 0$: repeated root -2, so CF $= (A + Bx)e^{-2x}$. Since e^{-2x} and xe^{-2x} solve the homogeneous equation, try PI $= \frac{\lambda x^2}{2}e^{-2x}$: with $y = \frac{\lambda x^2}{2}e^{-2x}$, $y' = \lambda(x - x^2)e^{-2x}$ and $y'' = \lambda(2x^2 - 4x + 1)e^{-2x}$. Combining: $y'' + 4y' + 4y = \lambda e^{-2x}\left[(2x^2 - 4x + 1) + 4(x - x^2) + 2x^2\right] = \lambda e^{-2x}$. So $\lambda = 1$ and the general solution is $y = (A + Bx)e^{-2x} + \frac{x^2}{2}e^{-2x}$.

Exercise 3.2

3.2 *(Edexcel IAL WFM02 June 2025 Q4, condensed.)* Given that $y = \lambda x e^{3x}$ is a particular integral of the differential equation $4\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 3y = 78e^{3x}$, (a) determine the value of λ. (b) Hence determine the general solution of the differential equation. Given also that $y = \frac{9}{2}$ and $\dfrac{dy}{dx} = 0$ at $x = 0$, (c) determine the particular solution. [6]

PI: $y = \lambda x e^{3x}$

$x = 0$: $y = \frac{9}{2}$, $\dfrac{dy}{dx} = 0$

Exercise 3.2

3.2 *(Edexcel IAL WFM02 June 2025 Q4, condensed.)* Given that $y = \lambda x e^{3x}$ is a particular integral of the differential equation $4\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 3y = 78e^{3x}$, (a) determine the value of λ. (b) Hence determine the general solution of the differential equation. Given also that $y = \frac{9}{2}$ and $\dfrac{dy}{dx} = 0$ at $x = 0$, (c) determine the particular solution. [6]

PI: $y = \lambda x e^{3x}$

$x = 0$: $y = \frac{9}{2}$, $\dfrac{dy}{dx} = 0$

3.2 (a) $y' = \lambda(1 + 3x)e^{3x}$, $y'' = \lambda(6 + 9x)e^{3x}$, so $4y'' - 11y' - 3y = \lambda e^{3x}(24 + 36x - 11 - 33x - 3x) = 13\lambda e^{3x} = 78e^{3x}$, hence $\lambda = 6$. (b) $4m^2 - 11m - 3 = (4m + 1)(m - 3)$, so $m = -\frac{1}{4}$ or 3: $y = Ae^{-x/4} + Be^{3x} + 6xe^{3x}$. (c) $y(0) = A + B = \frac{9}{2}$; $y' = -\frac{A}{4}e^{-x/4} + 3Be^{3x} + 6(1 + 3x)e^{3x}$, so $y'(0) = -\frac{A}{4} + 3B + 6 = 0$. From the first, $A = \frac{9}{2} - B$; substituting: $-\frac{1}{4}(\frac{9}{2} - B) + 3B = -6$, so $-\frac{9}{8} + \frac{13B}{4} = -6$, $\frac{13B}{4} = -\frac{39}{8}$, $B = -\frac{3}{2}$, and $A = 6$. Particular solution: $y = 6e^{-x/4} - \frac{3}{2}e^{3x} + 6xe^{3x}$.

Local review guidance: correct IF or auxiliary treatment; trial PI adjusted for resonance when the plain trial dies; constants found from the full general solution, not the CF. This is authored classroom guidance.