Course Companion

AQA · GCSE · Mathematics

Every topic handout in one volume

全部专题讲义合集

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1 · Number

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Number: the foundations everything else stands on

Why it matters. In August 1977 the magazine *Scientific American* printed a coded message and offered 100 US dollars to anyone who could read it. The code, invented that year by Ron Rivest, Adi Shamir and Leonard Adleman, used a 129-digit number made by multiplying two large prime numbers together. Multiplying the primes takes a computer a fraction of a second; finding them again from their product is enormously harder. The message stayed secret for 17 years, until 1994, when 600 volunteers with 1,600 computers finally found the two primes. The message read "The magic words are squeamish ossifrage." The same idea, now with much larger primes, protects every card payment and every secure website you use. This topic is about numbers: primes and factors, powers and roots, fractions and percentages, and how accurate an answer really is.

In this unit you will:

- calculate with integers, decimals, fractions and negative numbers, using the correct order of operations;
- write numbers as products of prime factors, and use them to find the HCF and LCM;
- use the laws of indices, standard form and (higher tier) surds;
- convert between fractions, decimals and percentages, and use percentages as multipliers;
- round, estimate and (higher tier) work with error intervals and bounds.

AQA GCSE Mathematics (8300), Number (N1 to N16). Number skills are tested on all three papers, at Foundation and Higher tier; content marked (H) is Higher tier only.

Calculating with numbers, factors and primes

Order and signs. Use the symbols $=$, $\neq$, $<$, $>$, $\leq$ and $\geq$ to compare numbers, including negative numbers, decimals and fractions. On a number line, -7 is less than -2 because it is further to the left.

The four operations. You must be able to add, subtract, multiply and divide integers, decimals and fractions (proper, improper and mixed numbers) without a calculator, often in money problems about profit, loss, interest, tax and VAT.

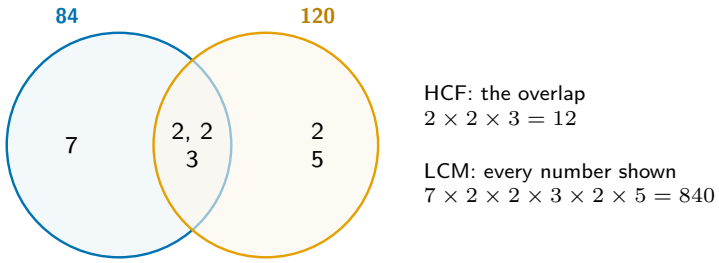
Order of operations. Work out brackets first, then indices (powers and roots), then multiplication and division, then addition and subtraction. This order is often remembered as **BIDMAS** 运算顺序. For example, $3+4\times2^2 = 3+4\times4 = 3+16 = 19$, not 28^2 or 196.

Factors, multiples and primes. A factor of 12 divides into it exactly: 1, 2, 3, 4, 6, 12. A multiple of 12 is in its times table: 12, 24, 36, … A prime number has exactly two factors, 1 and itself: 2, 3, 5, 7, 11, 13, … (1 is not prime). Every whole number greater than 1 can be written as a product of primes in exactly one way. This is its **prime factorisation** 质因数分解; for example, $200 = 2^3 \times 5^2$.

The highest common factor (HCF) of two numbers is the largest number that divides both; the lowest common multiple (LCM) is the smallest number that both divide into. From prime factorisations:

- HCF: multiply the primes they share, using the lower power of each;
- LCM: multiply every prime that appears, using the higher power of each.

Worked example 1. $84 = 2^2 \times 3 \times 7$ and $120 = 2^3 \times 3 \times 5$. HCF $= 2^2 \times 3 = 12$. LCM $= 2^3 \times 3 \times 5 \times 7 = 840$. Check: $12 \times 840 = 10\,080 = 84 \times 120$. (For any two numbers, HCF × LCM = the product of the numbers.)



The overlap gives the HCF; everything in the diagram gives the LCM.

Worked example 2. Buses leave a station every 12 minutes on route A and every 18 minutes on route B. Both leave at 9:00. When do they next leave together? We need the LCM of 12 and 18. $12 = 2^2 \times 3$ and $18 = 2 \times 3^2$, so LCM $= 2^2 \times 3^2 = 36$. They next leave together at 9:36.

Counting (H). If there are m ways to do one thing and n ways to do another, there are $m \times n$ ways to do both. A 4-digit PIN using the digits 0 to 9 has $10 \times 10 \times 10 \times 10 = 10\,000$ possibilities.

Indices, standard form and surds

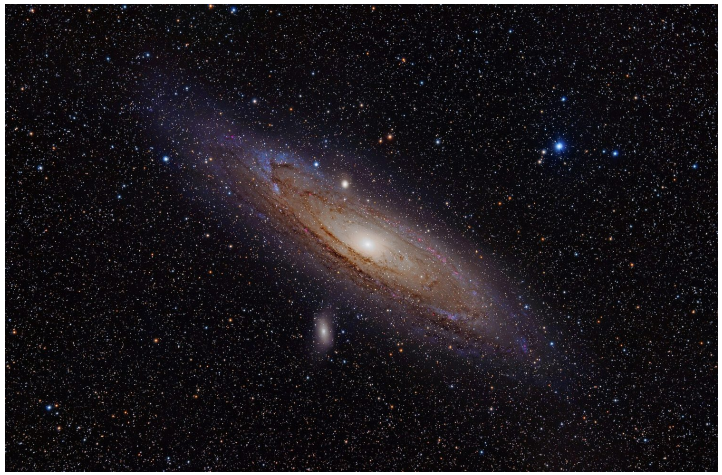
Laws of indices. A power, or index, tells you how many copies of a number are multiplied together: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. When the base is the same, the laws below let you combine powers without writing them out.

$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n} \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \text{ (H)}$$

Worked example 3 (H). Evaluate $27^{\frac{2}{3}}$ and $4^{-\frac{1}{2}}$. $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$. $4^{-\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$. Take the root first: it keeps the numbers small.

Standard form. Very large and very small numbers are written in **standard form** 标准形式 as $A \times 10^n$, where $1 \le A < 10$ and n is an integer. The distance to the Andromeda galaxy is about 2.4×10^{19} km; the width of a human hair is about 8×10^{-5} m.



A galaxy: distances this large are written compactly in standard form

To multiply, multiply the numbers and add the powers; to divide, divide the numbers and subtract the powers. Then make sure the first number is between 1 and 10.

Worked example 4. $(3 \times 10^7) \times (4 \times 10^{-3}) = 12 \times 10^4 = 1.2 \times 10^5$. To add or subtract, first write both numbers with the same power of 10: $(5 \times 10^4) + (3 \times 10^3) = (5 \times 10^4) + (0.3 \times 10^4) = 5.3 \times 10^4$.

Surds (H). A **surd** 根式 is a root that cannot be written exactly as a fraction, such as $\sqrt{2}$ or $\sqrt{12}$. Leaving it as a surd keeps the answer exact. Simplify by taking out square factors: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$. To rationalise a denominator, multiply the top and bottom so that the denominator no longer contains a root.

Worked example 5 (H).

$$\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3} \qquad \frac{1}{3+\sqrt{2}} = \frac{3-\sqrt{2}}{(3+\sqrt{2})(3-\sqrt{2})} = \frac{3-\sqrt{2}}{9-2} = \frac{3-\sqrt{2}}{7}$$

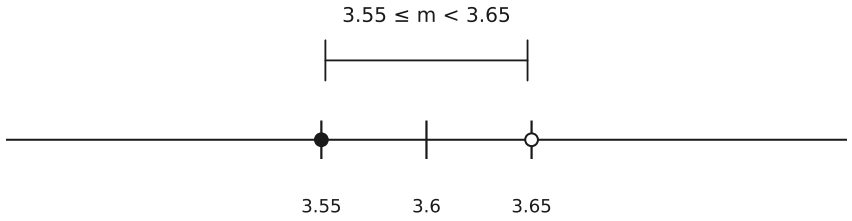
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Accuracy: rounding, estimating and bounds

Rounding. Round to a number of decimal places or significant figures, but only at the end: rounding in the middle of a calculation makes the final answer less accurate.

Estimating. Round every number to 1 significant figure and calculate. **Worked example 6.** Estimate $\frac{48.7\times3.12}{0.51}$. Rounded: $\frac{50\times3}{0.5} = \frac{150}{0.5} = 300$. (The exact answer is 297.9.)

Error intervals. A length given as 3.6 m to 1 decimal place could be anything from 3.55 m up to, but not including, 3.65 m. This is its **error interval** 误差区间, written $3.55 \le m < 3.65$. The **lower bound** 下界 is 3.55 and the **upper bound** 上界 is 3.65. If 3.6 was found by truncating (cutting off digits), the interval is $3.6 \le m < 3.7$.



m = 3.6 to 1 decimal place: closed at the lower bound, open at the upper

The error interval for 3.6 to one decimal place: closed at 3.55, open at 3.65.

Calculating with bounds (H). For positive numbers:

- the largest possible sum or product uses both upper bounds;
- the largest possible difference uses the upper bound minus the lower bound;
- the largest possible quotient uses the upper bound divided by the lower bound.

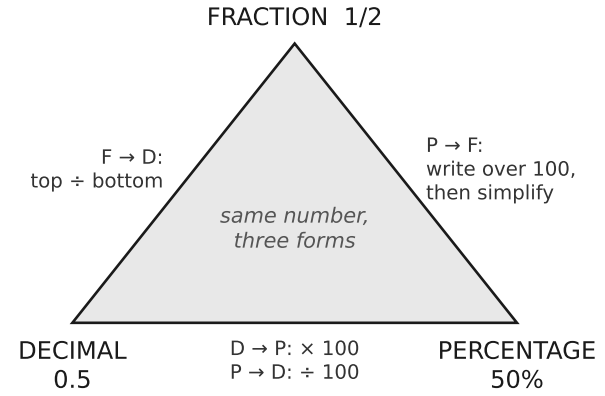
> **Worked example 7 (H).** A rectangle is 8 m by 5 m, each to the nearest metre. The bounds are 7.5 to 8.5 m and 4.5 to 5.5 m. The largest possible area is $8.5\times5.5 = 46.75$ m^2; the smallest is $7.5 \times 4.5 = 33.75$ m^2. The largest possible value of length ÷ width is $8.5 \div 4.5 = 1.89$ (3 s.f.), because dividing by the smallest width gives the largest answer.

> **Worked example 8 (H).** A car travels 120 km, to the nearest 10 km, in 1.5 hours, to the nearest 0.1 hour. Find the speed to a suitable accuracy. Upper bound $= 125 \div 1.45 = 86.2$ km/h; lower bound $= 115 \div 1.55 = 74.2$ km/h. These do not agree even to 1 significant figure, so the honest answer is "between 74 and 86 km/h": the measurements are not accurate enough to give a single value.

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Fractions, decimals and percentages

Fractions, decimals and percentages are three ways of writing the same number: $\frac{3}{8} = 0.375 = 37.5\%$.



The FDP triangle linking fraction, decimal and percentage.

Fraction calculations. To add or subtract, use a common denominator: $\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12} = 1\frac{5}{12}$. To multiply, multiply tops and bottoms. To divide, multiply by the reciprocal: $2\frac{1}{2} \div \frac{3}{4} = \frac{5}{2} \times \frac{4}{3} = \frac{20}{6} = 3\frac{1}{3}$.

Recurring decimals (H). A **recurring decimal** 循环小数 repeats forever, such as $0.\dot{6} = 0.666\ldots$. To turn it into a fraction:

> **Worked example 9 (H).** Let $x = 0.\dot{4}\dot{5} = 0.4545\ldots$. Two digits repeat, so multiply by 100: $100x = 45.4545\ldots$. Subtract: $99x = 45$, so $x = \frac{45}{99} = \frac{5}{11}$.

Percentages as multipliers. A **multiplier** 乘数 makes percentage change quick: an increase of 15% means multiplying by 1.15; a decrease of 20% means multiplying by 0.80. "$\frac{3}{5}$ of 40" means $40 \times \frac{3}{5} = 24$.

> **Worked example 10: reverse percentage.** In a sale, a coat costs 68 yuan after a 15% reduction. What was the original price? The sale price is 85% of the original, so the original $= 68 \div 0.85 = 80$ yuan. A common mistake is to add 15% of 68, which gives 78.20 yuan.

Units. Convert carefully between metric units of length, area and volume. Because area has two dimensions, 1 m^2 $= 100 \times 100 = 10\,000$ cm^2; for volume, 1 m^3 $= 1\,000\,000$ cm^3.

Sources for this lesson

- AQA, *GCSE Mathematics (8300) specification*, subject content section 3.1 (Number).
- M. Gardner, "Mathematical games: a new kind of cipher that would take millions of years to break", *Scientific American* (August 1977).
- D. Atkins, M. Graff, A. K. Lenstra and P. C. Leyland, "The magic words are squeamish ossifrage", *ASIACRYPT '94* (1995).
- NASA Science, "Messier 31 (the Andromeda galaxy)": distance about 2.5 million light years.

Common errors

- Doing operations from left to right instead of using BIDMAS: $3 + 4 \times 2 = 11$, not 14.
- Counting 1 as a prime number.
- Using the higher powers for the HCF, or the lower powers for the LCM.
- Leaving a standard form answer as 12×10^4; the first number must be from 1 to less than 10.
- Rounding in the middle of a calculation.
- Adding the percentage back on for a reverse percentage; divide by the multiplier instead.
- Writing 1 $\text{m}^2 = 100\ \text{cm}^2$; it is $10\,000\ \text{cm}^2$.

Key terms

- **Prime number**: a whole number with exactly two factors, 1 and itself.
- **Prime factorisation**: writing a number as a product of prime numbers.
- **HCF**: the highest common factor, the largest number that divides into two or more numbers.
- **LCM**: the lowest common multiple, the smallest number that two or more numbers divide into.
- **Standard form**: a number written as $A \times 10^n$ with $1 \leq A < 10$.
- **Surd**: a root that cannot be written as an exact fraction, such as $\sqrt{2}$.
- **Error interval**: the range of values a rounded number could have come from.
- **Multiplier**: the number you multiply by to make a percentage change.

Check yourself

1. Work out $20 - 12 \div 4 + 2^3$.
2. Write 360 as a product of prime factors, and find the HCF and LCM of 360 and 84.
3. Write 0.000 52 in standard form, and work out $(6 \times 10^5) \div (3 \times 10^{-2})$.
4. (H) Simplify $\sqrt{50} + \sqrt{18}$.
5. A mass is 7.4 kg to 1 decimal place. Write its error interval.
6. A phone costs 2,400 yuan after a 20% increase. What was the price before the increase?
7. (H) Write $0.\dot{3}\dot{6}$ as a fraction in its simplest form.

Answers: 1. $20-3+8 = 25$. 2. $360 = 2^3 \times 3^2 \times 5$; $84 = 2^2 \times 3 \times 7$; HCF $= 2^2 \times 3 = 12$; LCM $= 2^3 \times 3^2 \times 5 \times 7 = 2520$. 3. 5.2×10^{-4}; 2×10^7. 4. $5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$. 5. $7.35 \leq m < 7.45$. 6. $2400 \div 1.20 = 2000$ yuan. 7. $100x - x = 36$, so $x = \frac{36}{99} = \frac{4}{11}$.

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Algebra

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Algebra: letters that stand for numbers

Why it matters. Around the year 820, at the House of Wisdom in Baghdad, the scholar Muhammad ibn Musa al-Khwarizmi wrote a book about "completion and balancing". The Arabic word for completion, *al-jabr*, became our word "algebra", and his own name became the word "algorithm". He had no letters or symbols: every problem was written in words. One of his examples was "a square and ten roots equal thirty-nine", which we now write as $x^2 + 10x = 39$. He solved it by drawing a real square and adding the missing corner, the method we still call completing the square, and found $x = 3$. Today the same algebra sets the price of a phone plan, steers a satellite and trains a search engine. This topic teaches you to write, rearrange and solve algebra, and to see it as graphs.

In this unit you will:

- use algebraic notation, expand brackets and factorise expressions;
- rearrange formulae and (higher tier) prove identities and use function notation;
- draw and read straight-line, quadratic and other graphs;
- solve linear and quadratic equations, simultaneous equations and inequalities;
- find and use the nth term of a sequence.

AQA GCSE Mathematics (8300), Algebra (A1 to A25). Algebra is tested on all three papers at both tiers; content marked (H) is Higher tier only. The formulae sheet gives the quadratic formula, but every other result here must be remembered.

Expressions, formulae and identities

Notation. In algebra ab means $a \times b$, $3y$ means $y+y+y$, a^2 means $a \times a$, and $\frac{a}{b}$ means $a \div b$. Like terms have the same letters and powers, so $3x^2 + 5x - x^2 + 2x$ simplifies to $2x^2 + 7x$.

Words with exact meanings.

- An expression has no equals sign: $4x - 1$.
- An equation is true for particular values: $4x - 1 = 11$ is true only when $x = 3$.
- A formula connects quantities: $v = u + at$.
- An **identity** 恒等式 is true for every value of the letter, and uses the sign $\equiv$: $(x + 2)^2 \equiv x^2 + 4x + 4$.

Expanding and factorising. Expanding multiplies out brackets; factorising puts them back. For two brackets, multiply every term in the first by every term in the second.

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> **Worked example 1.** Expand $(2x - 3)(x + 4)$. $2x \times x = 2x^2$, $2x \times 4 = 8x$, $-3 \times x = -3x$, $-3 \times 4 = -12$. Total: $2x^2 + 5x - 12$.

> **Worked example 2.** Factorise $x^2 - 2x - 15$. Find two numbers that multiply to -15 and add to -2: they are -5 and 3. So $x^2 - 2x - 15 = (x - 5)(x + 3)$. The difference of two squares is a special case: $x^2 - 49 = (x - 7)(x + 7)$.

> **Worked example 3 (H).** Factorise $6x^2 - 7x - 3$. Multiply $6 \times -3 = -18$ and find two numbers with product -18 and sum -7: -9 and 2. Split the middle term: $6x^2 - 9x + 2x - 3 = 3x(2x - 3) + 1(2x - 3) = (3x + 1)(2x - 3)$.

Algebraic fractions (H). An **algebraic fraction** 代数分式 has letters in it. Simplify it by factorising the top and bottom and cancelling a common bracket: $\frac{x^2-9}{x^2+3x} = \frac{(x-3)(x+3)}{x(x+3)} = \frac{x-3}{x}$. You may cancel a whole factor, never single terms.

Changing the subject. Do the same thing to both sides until the letter you want stands alone. From $v = u + at$: subtract u, then divide by a, to get $t = \frac{v-u}{a}$.

Proof (H). To prove an identity, expand or simplify one side until it matches the other. Testing a few numbers never proves that something is true for all values.

> **Worked example 4 (H).** Prove that the sum of any three consecutive integers is a multiple of 3. Call them n, $n+1$ and $n+2$. Their sum is $3n+3 = 3(n+1)$, which is 3 times an integer.

Functions (H)

A function is a rule that turns each input into one output. In function notation, $f(x) = 3x - 5$ means "multiply by 3, then subtract 5", and $f(4) = 7$.

A **composite function** 复合函数 applies one function and then another. $fg(x)$ means $f(g(x))$: do g first. An **inverse function** 反函数 f^{-1} undoes f.

> **Worked example 5 (H).** $f(x) = 3x - 5$ and $g(x) = x^2$.

- $fg(2) = f(g(2)) = f(4) = 7$, but $gf(2) = g(f(2)) = g(1) = 1$: the order matters.
- For f^{-1}, write $y = 3x - 5$ and make x the subject: $x = \frac{y+5}{3}$. So $f^{-1}(x) = \frac{x+5}{3}$.
- Check: $f^{-1}(f(4)) = f^{-1}(7) = \frac{12}{3} = 4$.

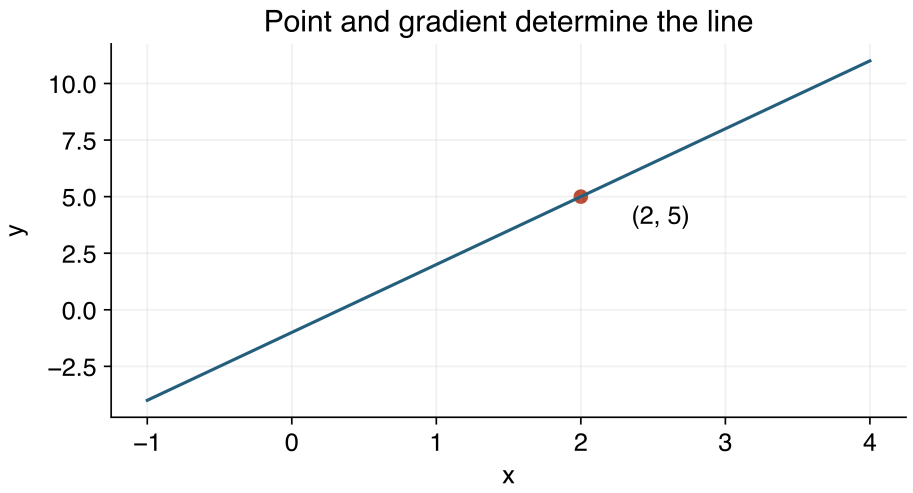
Straight-line graphs

Every straight line, except a vertical one, has an equation $y = mx + c$. The number m is the **gradient** 梯度, the steepness: how far y goes up when x increases by 1. The number c is the y-intercept, where the line crosses the y-axis. Between two points,

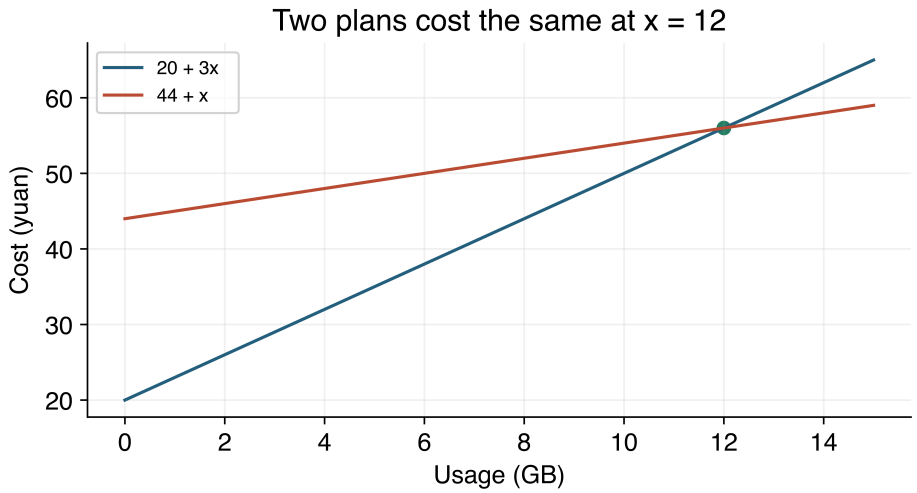
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Parallel lines have equal gradients. **Perpendicular lines** 垂直线 meet at a right angle, and their gradients multiply to -1 (H): a line with gradient 2 is perpendicular to a line with gradient $-\frac{1}{2}$.

> **Worked example 6.** Find the equation of the line through $(1, 7)$ and $(2, 9)$. $m = \frac{9-7}{2-1} = 2$. Substitute $(1, 7)$ into $y = 2x+c$: $7 = 2+c$, so $c = 5$ and the line is $y = 2x+5$. (H) The perpendicular line through $(1, 7)$ has gradient $-\frac{1}{2}$: $y - 7 = -\frac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + 7\frac{1}{2}$.



Gradient 3 through (2, 5): $y - 5 = 3(x - 2)$, so $y = 3x - 1$



The plans cost the same where $20 + 3x = 44 + x$: $x = 12$, cost 56 yuan

Real-life graphs. On a distance-time graph the gradient is the speed. On a velocity-time graph the gradient is the acceleration and the area under the graph is the distance travelled. When two cost lines cross, the plans cost the same: in the phone-plan graph above, $20 + 3x = 44 + x$ gives $x = 12$ GB, where both cost 56 yuan.

Quadratic and other graphs

The graph of a quadratic such as $y = x^2 - 10x + 21$ is a U-shaped curve called a parabola. It crosses the x-axis at the roots of $x^2 - 10x + 21 = 0$, it crosses the y-axis at $(0, 21)$, and it has a turning point on its line of symmetry. A hanging cable has nearly the same shape.

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A suspension bridge: the main cable hangs close to a parabola, the graph of a quadratic

Completing the square (H). Write $x^2 + bx + c$ as $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$. This **completing the square** 配方 shows the turning point at once.

> **Worked example 7 (H).** $y = x^2 - 6x + 5 = (x-3)^2 - 9 + 5 = (x-3)^2 - 4$. Because $(x-3)^2$ is never negative, the smallest value of y is -4, when $x = 3$. The turning point is $(3, -4)$ and the line of symmetry is $x = 3$.

Other graphs to recognise.

- Cubic, $y = x^3$: an S-shape.
- **Reciprocal** 倒数 graph, $y = \frac{1}{x}$: two separate curves; $x = 0$ is not allowed.
- (H) Exponential, $y = 2^x$: grows faster and faster and never touches the x-axis.
- (H) $y = \sin x$, $y = \cos x$ and $y = \tan x$: waves that repeat every 360° (180° for tan).

Transformations (H). $y = f(x) + a$ moves the graph of $y = f(x)$ up by a; $y = f(x + a)$ moves it left by a; $y = -f(x)$ reflects it in the x-axis; $y = f(-x)$ reflects it in the y-axis.

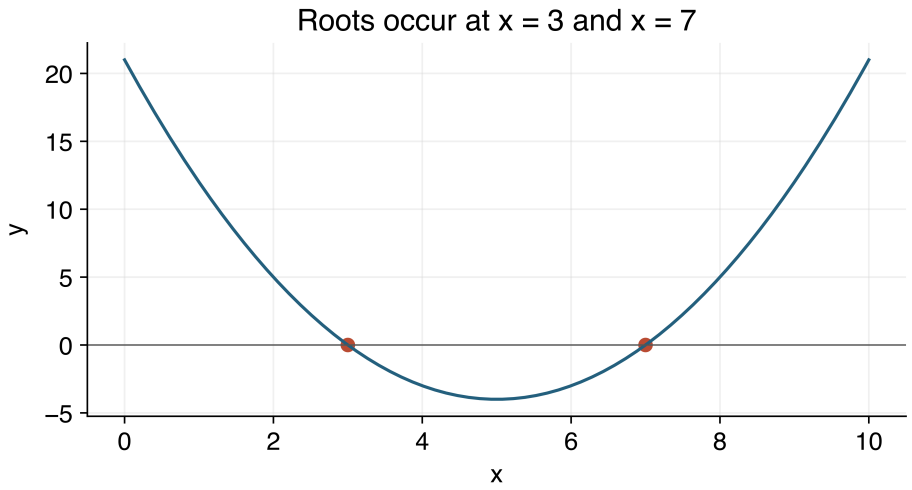
Circles (H). The circle with centre $(0, 0)$ and radius r has equation $x^2 + y^2 = r^2$. A tangent meets the radius at a right angle, so at the point $(3, 4)$ on $x^2 + y^2 = 25$ the radius has gradient $\frac{4}{3}$ and the tangent has gradient $-\frac{3}{4}$.

Solving equations

Linear equations. Collect the letters on one side and the numbers on the other. **Worked example 8.** $7x - 22 = 4x + 29$. Subtract $4x$: $3x - 22 = 29$. Add 22: $3x = 51$, so $x = 17$.

Quadratic equations by factorising. Make one side zero and factorise. If two brackets multiply to 0, one of them must be 0. Each value is a **root** 根 of the equation.

> **Worked example 9.** Solve $x^2 - 10x + 21 = 0$. $(x-3)(x-7) = 0$, so $x = 3$ or $x = 7$: exactly where the graph crosses the x-axis.



$x^2 - 10x + 21 = (x-3)(x-7)$: roots $x = 3$ and $x = 7$

The quadratic formula (H). For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** 判别式 $b^2 - 4ac$ tells you how many roots there are: two if it is positive, one if it is zero, none if it is negative.

> **Worked example 10 (H).** Solve $2x^2 - 5x - 4 = 0$, giving answers to 2 decimal places. $a = 2$, $b = -5$, $c = -4$. The discriminant is $25 + 32 = 57$, so there are two roots: $x = \frac{5 \pm \sqrt{57}}{4}$, so $x = 3.14$ or $x = -0.64$. Keep $\sqrt{57}$ exact until the last line.

Forming equations. Worked example 11. Ali is 4 years older than Ben, and Cara is twice Ben's age. Their ages add up to 44. Let Ben be x: $x + (x + 4) + 2x = 44$, so $4x = 40$ and $x = 10$. Ben is 10, Ali 14 and Cara 20. Always answer the question that was asked.

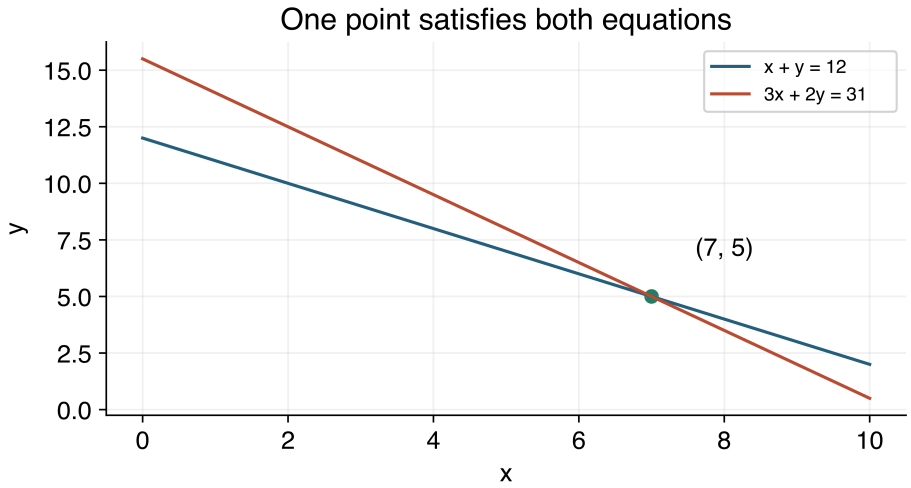
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Simultaneous equations and iteration

Two equations with two unknowns are solved together. On a graph, the solution is the point where the two lines cross.

Elimination. In **elimination** 消元法 you add or subtract the equations to remove one letter.

> **Worked example 12.** Solve $3x + 2y = 16$ and $5x - 2y = 8$. The y-terms have opposite signs, so add: $8x = 24$, $x = 3$. Substitute: $9 + 2y = 16$, so $y = 3.5$. If the signs are the same, subtract; if no terms match, multiply one equation first.



The solution is the crossing point: $x = 7$, $y = 5$ satisfies both equations

A line and a curve (H). Substitute the linear equation into the quadratic. **Worked example 13 (H).** $y = 2x + 5$ and $y = x^2$. Then $x^2 = 2x + 5$, so $x^2 - 2x - 5 = 0$, and the formula gives $x = 1 \pm \sqrt{6}$. There are two crossing points: $x = 3.45$, $y = 11.90$ and $x = -1.45$, $y = 2.10$ (2 d.p.).

Iteration (H). Iteration 迭代 finds an approximate root by repeating a formula. Rearrange the equation into the form $x = \ldots$, then put each answer back in.

> **Worked example 14 (H).** $x^3 - 3x - 1 = 0$ rearranges to $x = \sqrt[3]{3x + 1}$, so use $x_{n+1} = \sqrt[3]{3x_n + 1}$ with $x_1 = 1$. Then $x_2 = 1.587$, $x_3 = 1.793$, $x_4 = 1.855$, $x_5 = 1.872$, and the values settle towards the root $x = 1.879$ (3 d.p.).

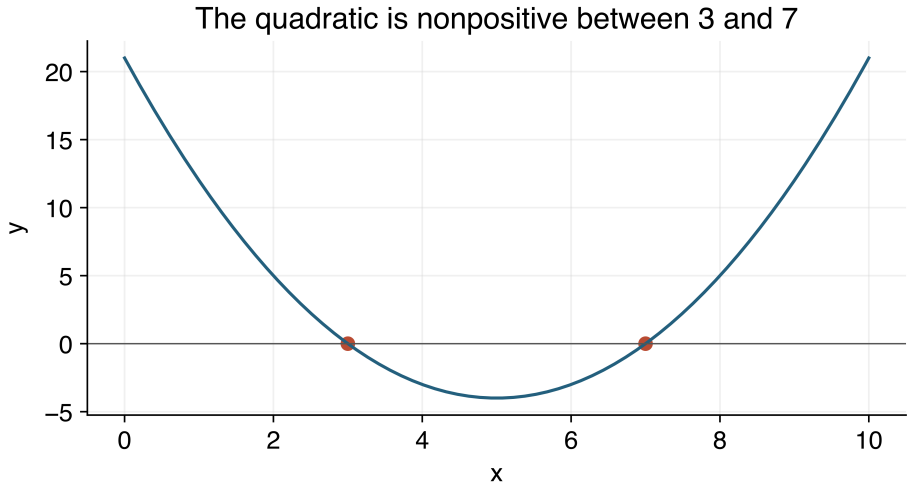
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Inequalities

Solve a linear inequality like an equation, but if you multiply or divide by a negative number, turn the sign round: from $-2x < 6$ you get $x > -3$. On a number line, an open circle means the end value is not included ($<$ or $>$) and a filled circle means it is ($\leq$ or $\geq$).

> **Worked example 15.** Find the integers n with $-3 < 2n + 1 \leq 9$. Subtract 1: $-4 < 2n \leq 8$. Divide by 2: $-2 < n \leq 4$. The integers are $-1, 0, 1, 2, 3, 4$.

Quadratic inequalities (H). Find the roots, then look at the graph. $x^2 - 10x + 21 \leq 0$ where the curve is on or below the x-axis, between the roots: $3 \leq x \leq 7$. For $x^2 - x - 6 > 0$, the roots are -2 and 3 and the curve is above the axis outside them: $x < -2$ or $x > 3$. In **set notation** 集合记号 this is $\{x : x < -2\} \cup \{x : x > 3\}$.



$x^2 - 10x + 21 = (x - 3)(x - 7) \leq 0$ for $3 \leq x \leq 7$

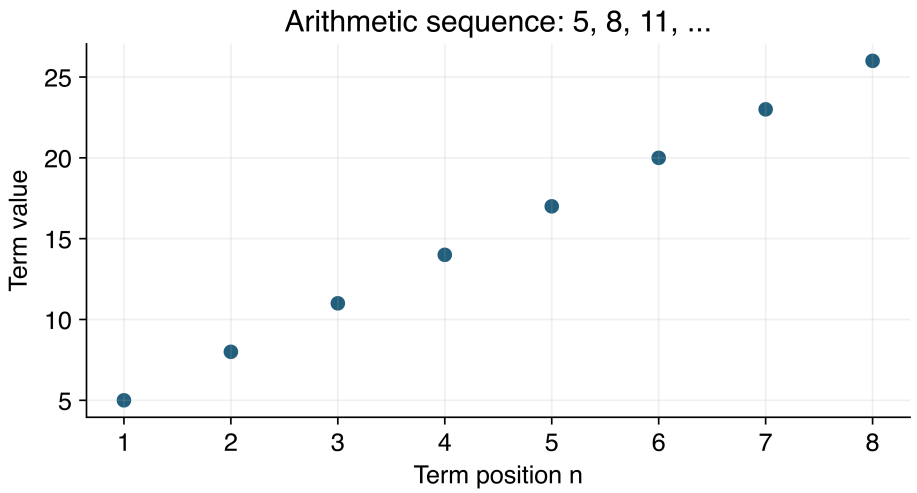
Regions (H). An inequality in x and y is a region on a graph. Draw the boundary as a solid line for $\leq$ or $\geq$ and as a dashed line for $<$ or $>$, then shade the correct side.

Sequences

A sequence is a list of numbers that follows a rule. A term-to-term rule says how to get the next term; a position-to-term rule, the **nth term** 通项, gives any term from its position n. You should know the square numbers (1, 4, 9, 16, …), cube numbers (1, 8, 27, …), triangular numbers (1, 3, 6, 10, …) and Fibonacci-type sequences, where each term is the sum of the two before.

Arithmetic sequences. An **arithmetic progression** 等差数列 adds the same

number each time. For 5, 8, 11, 14, … the difference is 3, so the nth term starts $3n$; since $3 \times 1 = 3$ and the first term is 5, the nth term is $3n + 2$. The eighth term is $3 \times 8 + 2 = 26$.



Add 3 each time: nth term 3n + 2, so the eighth term is 26

Geometric sequences. A **geometric progression** 等比数列 multiplies by the same number each time: 3, 6, 12, 24, … has ratio 2 and nth term $3 \times 2^{n-1}$.

Quadratic sequences (H). If the second differences are constant, the nth term is $an^2 + bn + c$ with a equal to half the second difference.

Worked example 16 (H). 5, 9, 17, 29, 45, … has first differences 4, 8, 12, 16 and second difference 4, so $a = 2$. Subtract $2n^2$ (2, 8, 18, 32, 50) to leave 3, 1, −1, −3, −5, which is $-2n + 5$. The nth term is $2n^2 - 2n + 5$. Check $n = 4$: $32 - 8 + 5 = 29$.

Sources for this lesson

- AQA, *GCSE Mathematics (8300) specification*, subject content section 3.2 (Algebra).
- Muhammad ibn Musa al-Khwarizmi, *The Compendious Book on Calculation by Completion and Balancing* (Baghdad, about 820), the problem $x^2 + 10x = 39$.
- AQA, *GCSE Mathematics* Higher Papers 1 and 3 (June 2024), the linear-equation and identity questions.

Common errors

- Writing $(x + 3)^2 = x^2 + 9$; it is $x^2 + 6x + 9$.
- Cancelling single terms in a fraction: $\frac{x+6}{x}$ is not 6.
- Doing f first in $fg(x)$; g is applied first.
- Forgetting to turn the inequality sign round after dividing by a negative number.
- Giving only one root of a quadratic, or one crossing point of a line and a curve.
- Rounding $\sqrt{57}$ before the final step.
- Writing the nth term of 5, 8, 11, … as $n + 3$; the difference 3 must multiply n.

Key terms

- **Identity**: a statement that is true for every value of the letter, written with $\equiv$.
- **Gradient**: the steepness of a line, the change in y for each increase of 1 in x.
- **Root**: a value of x that makes an equation true; where a graph crosses the x-axis.
- **Discriminant**: $b^2 - 4ac$, which tells you how many roots a quadratic has.
- **Elimination**: removing one unknown by adding or subtracting equations.
- **Iteration**: repeating a formula, using each answer as the next input.
- **nth term**: the rule that gives any term of a sequence from its position.

Check yourself

1. Expand and simplify $(x + 5)(x - 2)$.
2. Factorise $x^2 + 7x + 12$, and hence solve $x^2 + 7x + 12 = 0$.
3. Make r the subject of $A = \pi r^2$.
4. Find the equation of the line through $(0, -3)$ and $(2, 5)$.
5. Solve $2x + y = 11$ and $x - y = 1$.
6. (H) Write $x^2 + 8x + 3$ in the form $(x + p)^2 + q$ and give the turning point of $y = x^2 + 8x + 3$.
7. Find the nth term of 7, 11, 15, 19, … and the 20th term.

Answers: 1. $x^2 + 3x - 10$. 2. $(x + 3)(x + 4)$, so $x = -3$ or $x = -4$. 3. $r = \sqrt{\frac{A}{\pi}}$. 4. $m = \frac{8}{2} = 4$, so $y = 4x - 3$. 5. Adding gives $3x = 12$, so $x = 4$ and $y = 3$. 6. $(x + 4)^2 - 13$; turning point $(-4, -13)$. 7. $4n + 3$; the 20th term is 83.

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Ratio, proportion and rates of change

AQA · GCSE · Mathematics

Why it matters. On 23 September 1999 NASA's Mars Climate Orbiter fired its engine to go into orbit around Mars, and was never heard from again. It had flown far too low and was destroyed in the Martian atmosphere. The investigation found a simple cause. One team's software gave the push of the thrusters in pound-force seconds; the navigation team read the numbers as newton-seconds. One pound-force second is about 4.45 newton-seconds, so every small correction during the ten-month flight had been wrong by that factor. A mission that cost about 327 million US dollars was lost to a unit conversion. This topic is about comparing quantities: units and rates, ratios, percentages, proportion and growth.

In this unit you will:

- convert between units, including compound units such as speed, density and pressure;
- simplify ratios, share in a ratio and solve best-buy problems;
- use percentage multipliers, including reverse percentages and compound interest;
- solve direct and inverse proportion problems and (higher tier) form their equations;
- use scale factors for lengths, areas and volumes, and read a gradient as a rate of change.

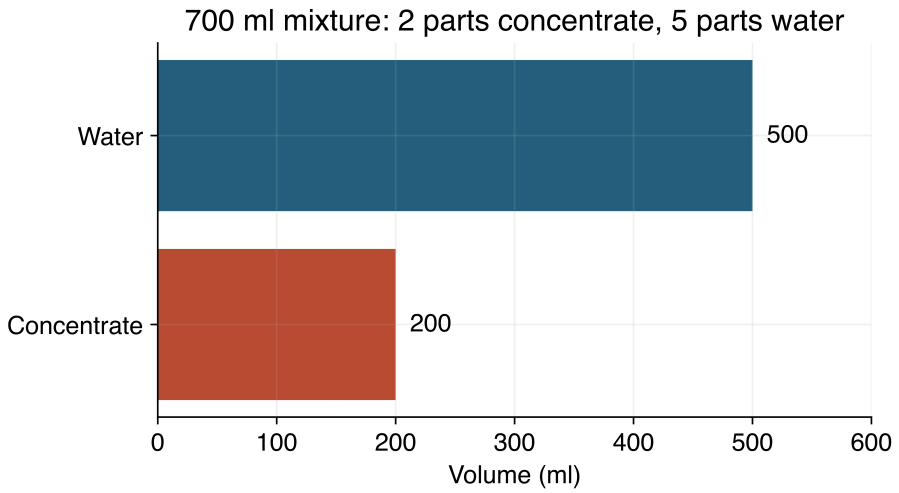
AQA GCSE Mathematics (8300), Ratio, proportion and rates of change (R1 to R16). These skills are tested on all three papers at both tiers; content marked (H) is Higher tier only. The formulae sheet gives nothing for this topic.

Ratio and sharing

A ratio compares two or more quantities. Simplify it by dividing every part by the same number: $45 : 60 = 3 : 4$. A ratio can also be written as a fraction: in a class with boys and girls in the ratio $3 : 4$, the boys are $\frac{3}{7}$ of the class, because there are 7 parts in the whole. This is a **part-to-whole ratio** 部分与整体的比.

> **Worked example 1.** A drink is made from concentrate and water in the ratio $2 : 5$. How much of each is in 700 ml? There are $2 + 5 = 7$ parts, so one part is $700 \div 7 = 100$ ml: 200 ml of concentrate and 500 ml of water.

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Ratio 2 : 5 has 7 parts; 700 ÷ 7 = 100 ml per part, so 200 ml and 500 ml

> **Worked example 2.** In a bag, red and blue counters are in the ratio $3 : 8$. There are 35 more blue counters than red. How many counters are there? The difference is $8 - 3 = 5$ parts, so one part is $35 \div 5 = 7$. There are $11 \times 7 = 77$ counters.

Proportion with ratios. Two ratios that are equal are in proportion. The unitary method finds the value of one unit first. If 6 pens cost 9 yuan, one pen costs 1.50 yuan and 10 pens cost 15 yuan.

Best buys. To compare value, find the price of the same amount.

> **Worked example 3.** A 2.5 kg bag of rice costs 64 yuan, and a 400 g box costs 11.20 yuan. Per kilogram, the bag costs $64 \div 2.5 = 25.60$ yuan and the box costs $11.20 \div 0.4 = 28$ yuan. The bag is the **best buy** 最优惠购买. Show both unit prices: the examiner wants the division, not just the choice.

Percentages

A percentage is a number of parts per hundred. The quickest way to change an amount by a percentage is a **multiplier** 乘数:

- an increase of 12% means $\times 1.12$;
- a decrease of 17% means $\times 0.83$;
- 120% of an amount means $\times 1.2$.



Shop discounts are percentage changes: a 20% reduction multiplies the price by 0.8

Percentage change.

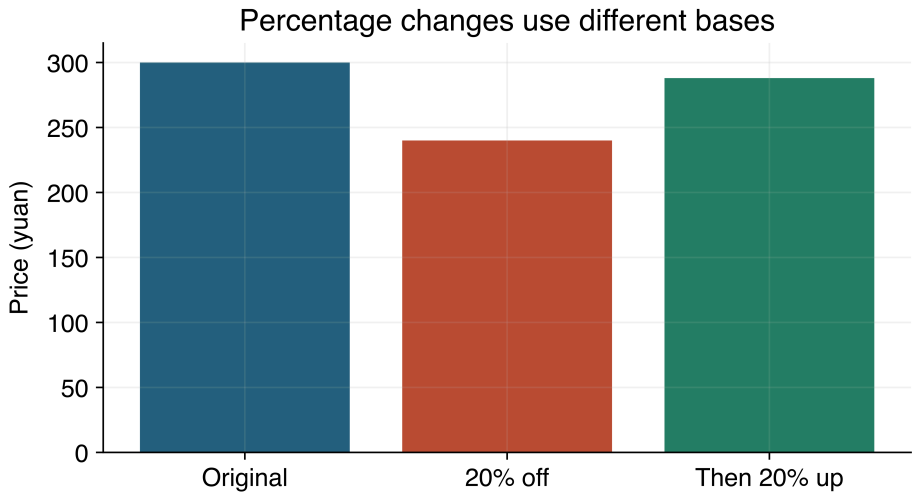
$$\text{percentage change} = \frac{\text{change}}{\text{original}} \times 100$$

If a price goes from 80 yuan to 92 yuan, the change is 12 and the percentage increase is $\frac{12}{80} \times 100 = 15\%$.

> **Worked example 4: reverse percentage.** After a fall of 12%, a town has 2200 people. What was the population before? The new population is 88% of the old one, so the old one is $2200 \div 0.88 = 2500$.

> **Worked example 5: one change after another.** A jacket costs 300 yuan. Its price is cut by 20% and later raised by 20%. The new price is $300 \times 0.8 \times 1.2 = 288$ yuan, not 300 yuan, because the second 20% is a percentage of the smaller price, 240 yuan.

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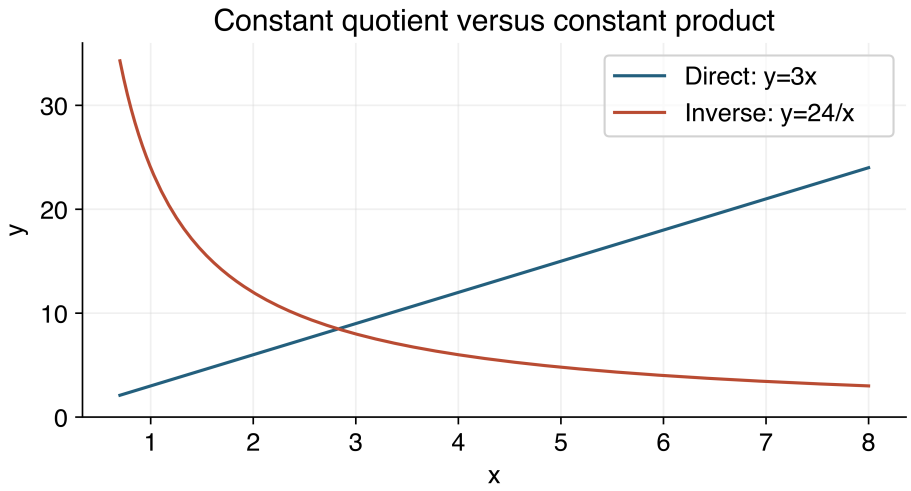


20% off then 20% up gives 288, not 300: the second 20% is of 240

Direct and inverse proportion

Two quantities are in **direct proportion** 正比例 when they increase at the same rate: double one and the other doubles. We write $y \propto x$, which means $y = kx$. The number k is the **constant of proportionality** 比例常数. The graph is a straight line through the origin.

Two quantities are in **inverse proportion** 反比例 when one doubles as the other halves: $y \propto \frac{1}{x}$, so $y = \frac{k}{x}$. The product xy stays the same. More workers finish a job in less time.



Direct proportion keeps $y \div x$ constant (3); inverse proportion keeps $x \times y$ constant (24)

Worked example 6. Three workers paint a fence in 8 hours. How long would 4 workers take, working at the same rate? The total work is $3 \times 8 = 24$ worker-hours, so 4 workers take $24 \div 4 = 6$ hours.

Forming equations (H). Write the proportion with k, use one pair of values to find k, then use the equation.

Worked example 7 (H). y is directly proportional to x^3, and $y = 24$ when $x = 2$. Find y when $x = 5$. $y = kx^3$, so $24 = 8k$ and $k = 3$. Then $y = 3 \times 125 = 375$.

Worked example 8 (H). The light p from a lamp is inversely proportional to the square of the distance d, and $p = 8$ when $d = 2$. $p = \frac{k}{d^2}$, so $k = 8 \times 4 = 32$ and $p = \frac{32}{d^2}$. At $d = 4$, $p = 2$: doubling the distance gives a quarter of the light.

Units and compound measures

Converting units. Learn the metric links: 1 km = 1000 m, 1 m = 100 cm, 1 kg = 1000 g, 1 litre = 1000 cm^3. Time is not decimal: 2 hours 36 minutes is $2 + \frac{36}{60} = 2.6$ hours, not 2.36 hours.

A compound unit combines two units, such as km/h or g/cm^3.

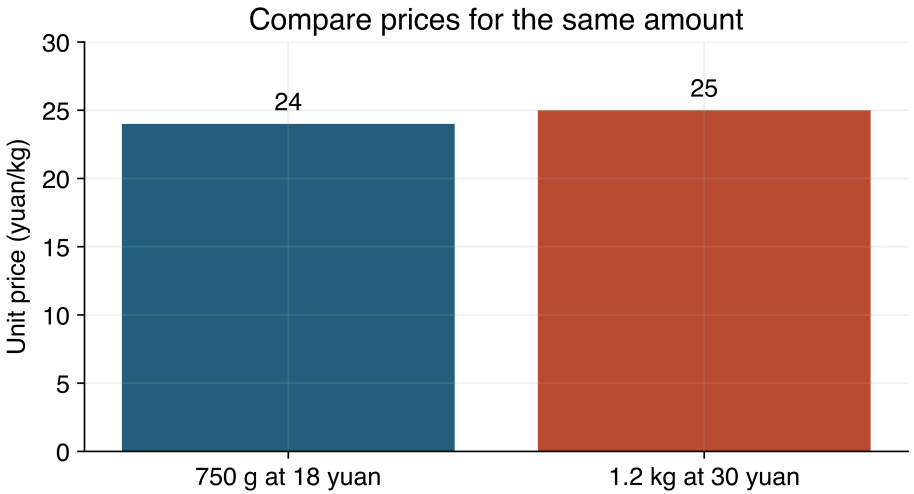
$$\text{speed} = \frac{\text{distance}}{\text{time}} \qquad \text{density} = \frac{\text{mass}}{\text{volume}} \qquad \text{pressure} = \frac{\text{force}}{\text{area}}$$

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Density 密度 tells you how much mass is packed into each unit of volume. **Pressure** 压强 is the force on each unit of area, measured in N/m^2.

Worked example 9. A coach travels 234 km in 2 hours 36 minutes. Its average speed is $234 \div 2.6 = 90$ km/h.

Worked example 10. A gold bar has mass 12.5 kg and volume 650 cm^3. Its density is $12\,500 \div 650 = 19.2$ g/cm^3 (3 s.f.). Change kg to g first, so that the units match.

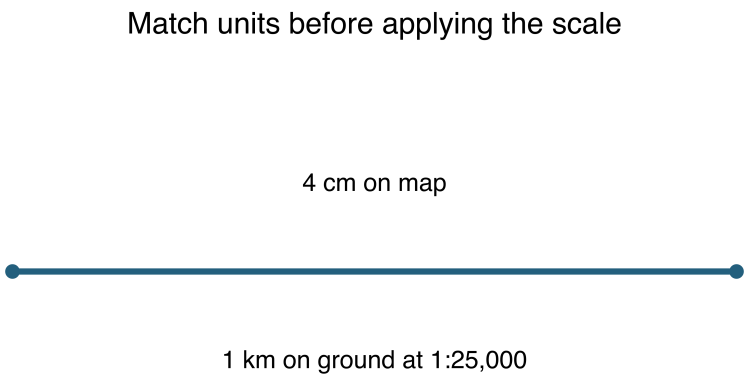


Compare the same amount: 18 ÷ 0.75 = 24 yuan/kg beats 30 ÷ 1.2 = 25 yuan/kg

Worked example 11. A box weighing 600 N stands on a face 0.5 m by 0.4 m. The area is 0.2 m^2, so the pressure is $600 \div 0.2 = 3000$ N/m^2. On a smaller face the same weight gives a larger pressure.

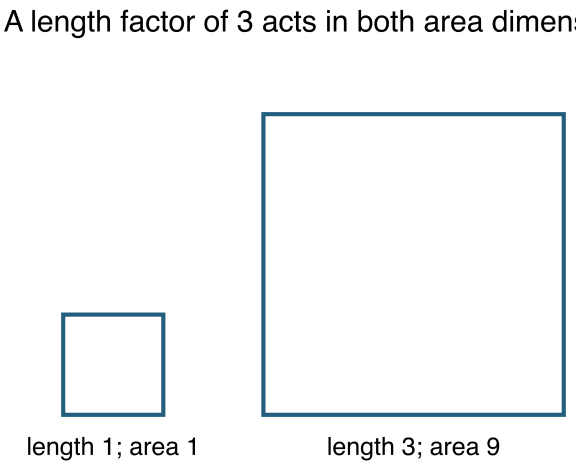
Scale, similarity and rates of change

Scale drawings. A map **scale** 比例尺 of 1 : 25 000 means 1 cm on the map is 25 000 cm in real life. A 4 cm line on the map is $4 \times 25\,000 = 100\,000$ cm, which is 1 km.



4 cm × 25 000 = 100 000 cm = 1 km

Length, area and volume. If two shapes are similar with a length scale factor of k, the **area scale factor** 面积比例因子 is k^2 and the volume scale factor is k^3. A square with sides 3 times as long has 9 times the area.



A length factor of 3 multiplies area by 3 × 3 = 9

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> **Worked example 12 (H).** Two similar statues have heights in the ratio 3 : 5. The smaller has volume 216 cm^3. The volume ratio is $3^3 : 5^3 = 27 : 125$, so the larger has volume $216 \times \frac{125}{27} = 1000$ cm^3.

Gradient as a rate. The gradient of a graph is a **rate of change** 变化率: how fast one quantity changes compared with another. On a distance-time graph it is the speed; on a velocity-time graph it is the acceleration.

Rates on a curve (H). On a curve the rate changes from point to point. The average rate between two points is the gradient of the chord joining them. The rate at one instant is the **tangent gradient** 切线斜率: draw a tangent touching the curve at that point and find its gradient as $\frac{\text{rise}}{\text{run}}$.

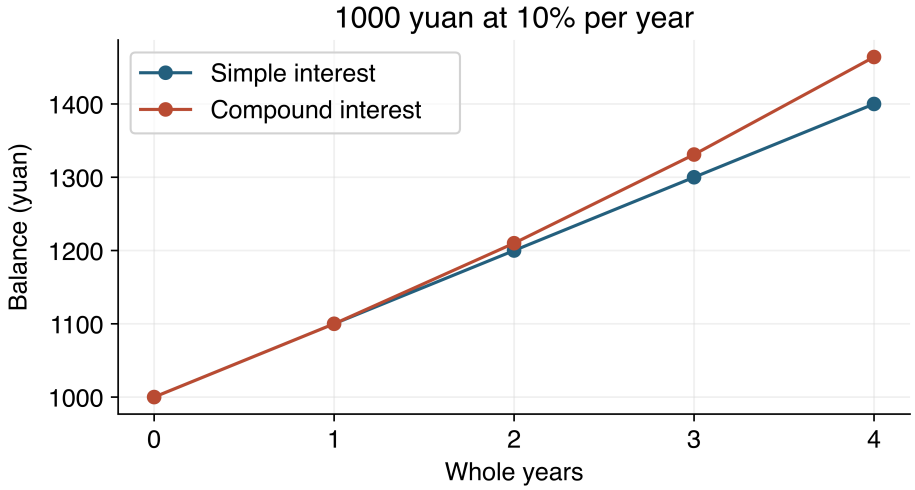
Growth and decay

Simple interest pays the same amount every year. **Compound interest** 复利 pays interest on the interest too, so the amount is multiplied by the same number every year:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where P is the starting amount, r the percentage rate and n the number of years. For decay, such as a car losing value, use $1 - \frac{r}{100}$.

> **Worked example 13.** 1000 yuan is saved at 10% a year for 4 years. Simple interest gives $1000 + 4 \times 100 = 1400$ yuan. Compound interest gives $1000 \times 1.1^4 = 1464.10$ yuan.



Simple interest adds 100 each year; compound interest grows by 10% of the new balance

Worked example 14. A car worth 120 000 yuan loses 15% of its value each year. After 3 years it is worth $120\,000 \times 0.85^3 = 73\,695$ yuan.

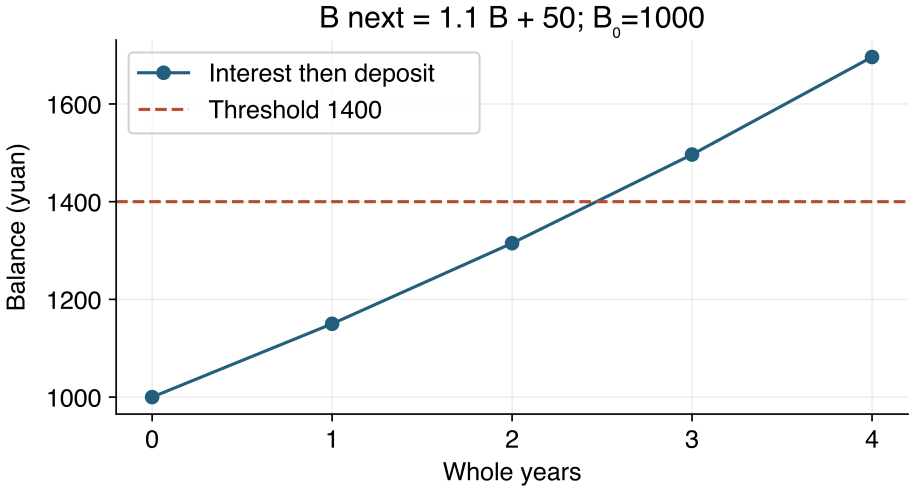
Worked example 15. A flat worth 900 000 yuan rises in value by 3% one year and 8% the next. Is it now worth more than 1 million yuan? $900\,000 \times 1.03 \times 1.08 = 1\,001\,160$ yuan, so yes, just.



A growing population: each year multiplies the last by the same factor, which is compound growth

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Iterative processes (H). An **iterative process** 迭代过程 uses the last value to find the next one. If 1000 yuan earns 10% a year and 50 yuan is added at the end of each year, then $B_{n+1} = 1.1B_n + 50$. The balances are 1000, 1150, 1315, 1496.50, so the balance first passes 1400 yuan after 3 years.



$B_{n+1} = 1.1B_n + 50$: 1000, 1150, 1315, 1496.5, so it first passes 1400 after 3 years

Sources for this lesson

- AQA, *GCSE Mathematics (8300) specification*, subject content section 3.3 (Ratio, proportion and rates of change).
- NASA, *Mars Climate Orbiter Mishap Investigation Board Phase I Report* (10 November 1999).
- AQA, *GCSE Mathematics* Higher Paper 2 (June 2024), the reverse-percentage and house-price questions, adapted.

Common errors

- Sharing in the ratio 2 : 5 by dividing by 5 instead of by the total number of parts, 7.
- Adding the percentage back on in a reverse percentage; divide by the multiplier instead.
- Thinking a 20% fall followed by a 20% rise returns to the start.
- Writing 2 hours 36 minutes as 2.36 hours.
- Using the length scale factor for an area or a volume.

- Mixing units, such as kg and cm^3, when finding density.
- Using simple interest when a question says "compound".

Key terms

- **Ratio**: a comparison of the sizes of two or more quantities, such as $3 : 4$.
- **Multiplier**: the number you multiply by to make a percentage change.
- **Direct proportion**: $y = kx$; both quantities change by the same factor.
- **Inverse proportion**: $y = \frac{k}{x}$; when one doubles, the other halves.
- **Density**: mass divided by volume.
- **Pressure**: force divided by area.
- **Compound interest**: interest added to the amount each year, so that later interest is paid on it too.
- **Rate of change**: how fast one quantity changes compared with another; the gradient of a graph.

Check yourself

1. Share 560 yuan in the ratio $3 : 5$.
2. A 750 g pack costs 18 yuan and a 1.2 kg pack costs 30 yuan. Which is better value?
3. A coat costs 425 yuan after a 15% reduction. What was the original price?
4. 5 machines fill an order in 12 hours. How long would 6 machines take?
5. A metal block has mass 540 g and volume 200 cm^3. Find its density.
6. 2000 yuan is invested at 3% compound interest for 4 years. Find the final amount.
7. (H) y is inversely proportional to x, and $y = 6$ when $x = 4$. Find y when $x = 3$.

Answers: 1. One part is $560 \div 8 = 70$, so 210 yuan and 350 yuan. 2. 24 yuan/kg against 25 yuan/kg, so the 750 g pack. 3. $425 \div 0.85 = 500$ yuan. 4. $60 \div 6 = 10$ hours. 5. $540 \div 200 = 2.7$ g/cm^3. 6. $2000 \times 1.03^4 = 2251.02$ yuan. 7. $y = \frac{24}{x}$, so $y = 8$.

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Geometry and measures

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Why it matters. In about 240 BC, Eratosthenes, the librarian at Alexandria in Egypt, measured the size of the whole Earth with a stick, a shadow and one geometry fact. He had heard that at noon on midsummer's day the Sun shone straight down a well at Syene, far to the south, so there were no shadows. At the same moment in Alexandria, a vertical stick cast a shadow at an angle of about 7.2°. Because the Sun's rays are parallel, alternate angles told him that the two cities were also 7.2° apart around the centre of the Earth, which is $\frac{1}{50}$ of a full turn. The cities were about 5000 stadia apart, so the Earth's circumference was about $50 \times 5000 = 250\,000$ stadia, close to the true value of about 40 000 km. Angles, circles, areas, triangles and vectors are the tools of this topic, and they still measure buildings, maps and planets.

In this unit you will:

- use angle facts, polygon properties, constructions and loci;
- describe transformations and use congruence and similarity;
- calculate perimeters, areas, volumes, arcs and sectors;
- use Pythagoras' theorem and trigonometry, including (higher tier) the sine and cosine rules and 3D problems;
- (higher tier) prove circle theorems and use vectors in proofs.

AQA GCSE Mathematics (8300), Geometry and measures (G1 to G25). This topic is tested on all three papers at both tiers; content marked (H) is Higher tier only. The formulae sheet gives the area of a trapezium, the volume of a prism, the circle formulae, Pythagoras' theorem, the trigonometric ratios, the sine and cosine rules and $\frac{1}{2}ab\sin C$; you must remember everything else, including arc and sector formulae, cone and sphere formulae, exact trigonometric values and the circle theorems.

Angles and polygons

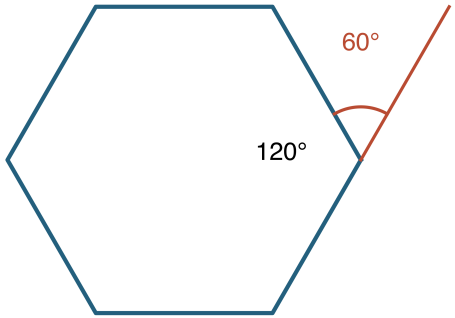
Angle facts. Every angle step in an answer needs a reason in words.

- Angles at a point add up to 360°; angles on a straight line add up to 180°.
- Vertically opposite angles are equal.
- Where a line crosses two parallel lines, alternate angles are equal, **corresponding angles** 同位角 are equal, and co-interior angles add up to 180°. Write "alternate angles are equal", never "Z-angles".
- The angles in a triangle add up to 180°.

Polygons. A polygon with n sides can be split into $n - 2$ triangles from one corner, so its interior angles add up to $(n-2) \times 180°$. A pentagon gives $3 \times 180° = 540°$. In a **regular polygon** 正多边形 all sides and all angles are equal, and each exterior angle is $\frac{360°}{n}$.

Worked example 1. Find the interior angle of a regular hexagon. Exterior angle $= 360° \div 6 = 60°$, so interior angle $= 180° - 60° = 120°$. Going the other way, a regular polygon with exterior angles of $45°$ has $360 \div 45 = 8$ sides.

Regular hexagon: interior 120°, exterior 60°

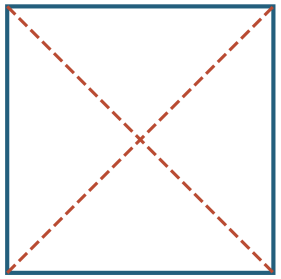


Regular hexagon: exterior angle $360° \div 6 = 60°$, interior angle $180° - 60° = 120°$

Quadrilaterals. A square has four equal sides and four right angles. A rectangle has opposite sides equal and four right angles. A rhombus has four equal sides, and its diagonals cross at right angles. A parallelogram has opposite sides equal and parallel. A kite has two pairs of equal adjacent sides. A trapezium has one pair of parallel sides. A **line of symmetry** 对称轴 divides a shape into two mirror halves: a square has four, but a rectangle that is not a square has only two, because its diagonals are not lines of symmetry.

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Square diagonals are symmetry axes



Square



Rectangle

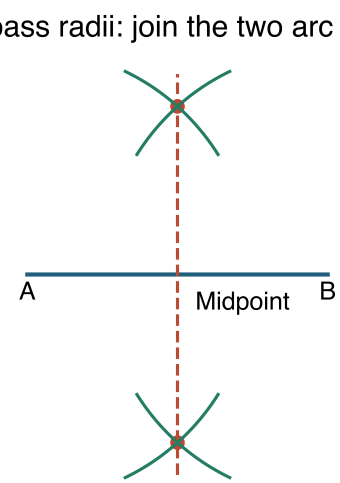
A square's diagonals are lines of symmetry; a rectangle's are not

Constructions, loci and transformations

Constructions. With a ruler and compasses you can construct:

- the **perpendicular bisector** 垂直平分线 of a line segment, which cuts it in half at right angles;
- the bisector of an angle;
- the perpendicular from a point to a line;
- an angle of 60°, from an equilateral triangle.

Leave the construction arcs on your drawing: they are the evidence the examiner looks for.

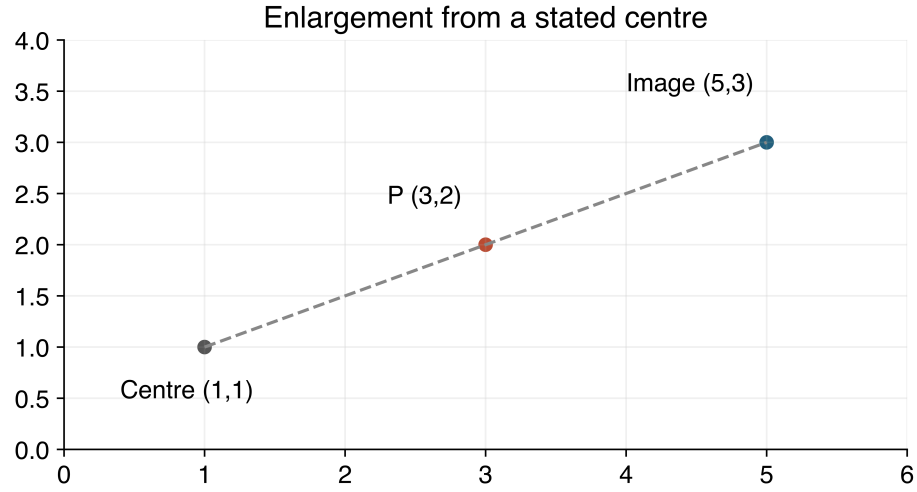


Equal compass arcs from A and B cross twice; the line through the crossings is the perpendicular bisector

Loci. A **locus** 轨迹 is the set of all points that follow a rule. Points a fixed distance from a point lie on a circle. Points the same distance from two points lie on the perpendicular bisector. Points the same distance from two lines lie on the angle bisector. The shortest distance from a point to a line is along the perpendicular.

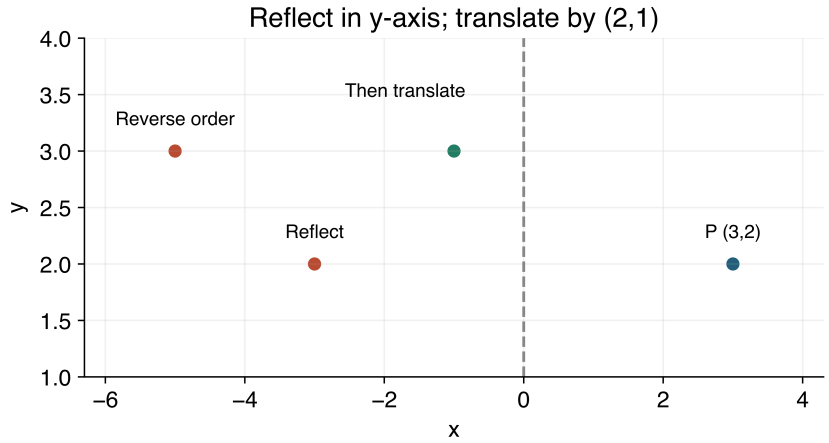
Transformations. A translation slides a shape by a column vector, such as $\begin{pmatrix}2\\1\end{pmatrix}$ (2 right, 1 up). A reflection needs a mirror line; a rotation needs a centre, an angle and a direction; an **enlargement** 放大 needs a centre and a scale factor. A scale factor of $\frac{1}{2}$ makes the shape smaller, and (H) a negative scale factor puts the image on the other side of the centre, upside down.

> **Worked example 2.** Enlarge $P(3,2)$ from the centre $C(1,1)$. P is 2 right and 1 up from C. With scale factor 2, the image is 4 right and 2 up from C, at $(5,3)$. With scale factor $\frac{1}{2}$ it is at $(2,1.5)$. (H) With scale factor -2, it is 4 left and 2 down from C, at $(-3,-1)$.



Scale factor 2 from (1, 1): the image is twice as far from the centre as P

Combining transformations (H). The order matters. Reflecting $P(3,2)$ in the y-axis and then translating by $\begin{pmatrix}2\\1\end{pmatrix}$ gives $(-1,3)$; translating first and then reflecting gives $(-5,3)$. Reflections, rotations and translations keep lengths and angles the same: these are **invariant** 不变量 properties. Two reflections in parallel mirror lines give a translation: reflecting $(3,2)$ in $x=0$ and then in $x=2$ gives $(7,2)$, a translation of 4 to the right.



Order matters: reflect then translate gives (−1, 3); translate then reflect gives (−5, 3)

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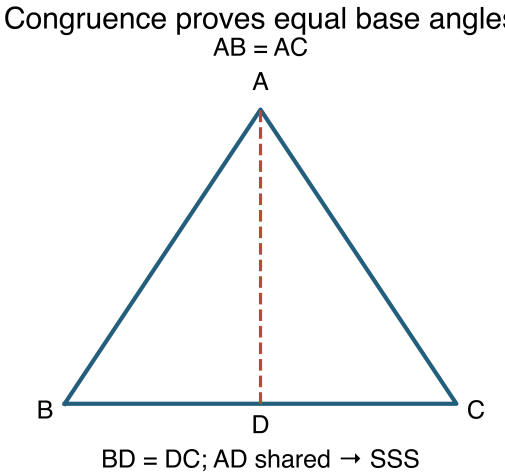
Congruence, similarity and circles

Congruence. Two shapes are congruent when they are exactly the same shape and size. To prove the **congruence** 全等 of two triangles, show that they match by one of four conditions:

- SSS: three sides equal;
- SAS: two sides and the angle between them;
- ASA: two angles and a side in the same position;
- RHS: a right angle, the hypotenuse and one other side.

Three equal angles (AAA) show only that triangles are similar, not congruent.

> **Worked example 3: a proof.** Triangle ABC has $AB = AC$, and D is the **midpoint** 中点 of BC. Prove that the base angles are equal. In triangles ABD and ACD: $AB = AC$ (given), $BD = DC$ (D is the midpoint) and AD is shared. So the triangles are congruent (SSS), and angle ABD = angle ACD.

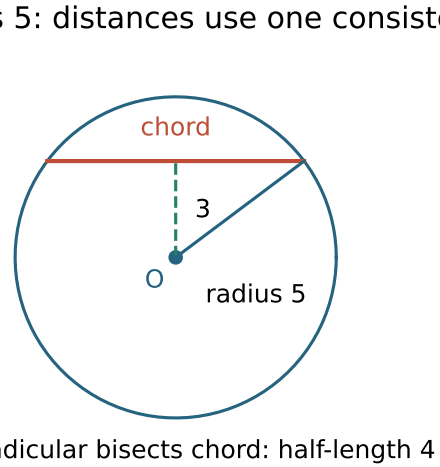


$AB = AC$, $BD = DC$ and AD shared: SSS congruence, so the base angles are equal

Similarity. Similar shapes have equal angles and their sides in the same ratio, the **scale factor** 相似比. If the length scale factor is k, areas scale by k^2 and (H) volumes by k^3.

Parts of a circle. The **circumference** 圆周长 is the distance around a circle. A chord is a straight line across a circle; a tangent touches it at one point. A **sector** 扇形 is a "pizza slice" between two radii and an arc. A **segment** 弓形 is the region between a chord and an arc.

> **Worked example 4.** A chord is 3 cm from the centre of a circle of radius 5 cm. How long is it? The perpendicular from the centre cuts the chord in half, making a right-angled triangle with hypotenuse 5 and one side 3. Half the chord is $\sqrt{5^2 - 3^2} = 4$, so the chord is 8 cm.



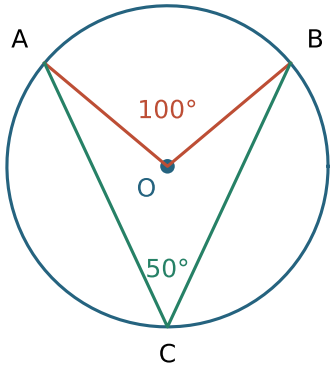
Radius 5, distance 3: half the chord is $\sqrt{(5^2 - 3^2)} = 4$, so the chord is 8

Circle theorems (H).

1. The **angle at the centre** 圆心角 is twice the angle at the circumference standing on the same arc.
2. The angle in a semicircle is 90°.
3. Angles in the same segment are equal.
4. Opposite angles of a **cyclic quadrilateral** 圆内接四边形 add up to 180°.
5. A tangent is perpendicular to the radius at the point where it touches.
6. The two tangents from an outside point are equal in length.
7. The perpendicular from the centre to a chord cuts the chord in half.
8. The **alternate segment theorem** 弦切角定理: the angle between a tangent and a chord equals the angle in the alternate segment.

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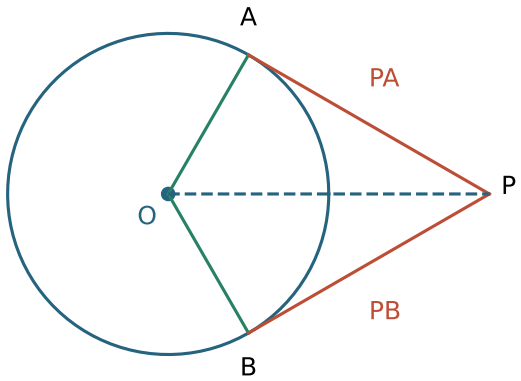
Angles on the same upper arc: centre = twice circumference



The angle at the centre, 100°, is twice the angle at the circumference, 50°

Proof of theorem 1 (H). Join the point C on the circumference to the centre O and continue the line. $OA = OC = OB$ (radii), so triangles OAC and OBC are isosceles. If the base angles are α and β, the exterior angles at O are 2α and 2β. So angle $AOB = 2(\alpha + \beta) = 2\times$ angle ACB.

Shared OP and equal radii → RHS → PA = PB



Equal radii, shared OP and right angles give congruent triangles (RHS), so PA = PB

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Worked example 5 (H). Two tangents from P touch a circle, centre O, at A and B, and angle $APB = 24°$. Find the angle ACB at a point C on the major arc. Angles OAP and OBP are 90° (tangent and radius), so in quadrilateral $OAPB$, angle $AOB = 360° - 90° - 90° - 24° = 156°$. The angle at the circumference is half of this: 78°.

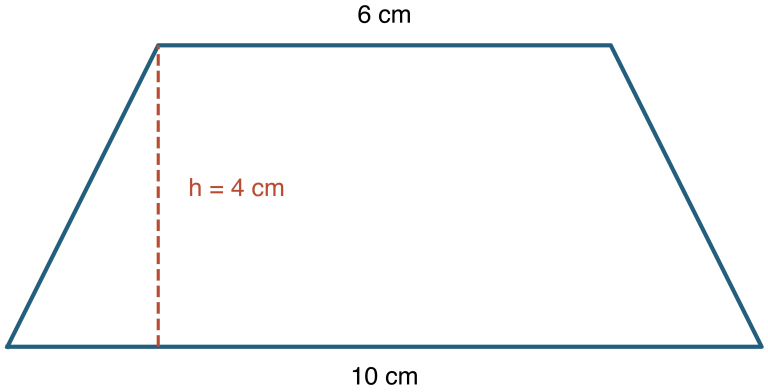
Perimeter, area and volume

Area formulae.

- Triangle: $\frac{1}{2} \times b \times h$, where h is the **perpendicular height** 垂直高度.
- Parallelogram: $b \times h$.
- Trapezium: $\frac{1}{2}(a + b)h$, where a and b are the parallel sides.
- Circle: $A = \pi r^2$, and the circumference is $C = 2\pi r = \pi d$.

Worked example 6. A trapezium has parallel sides 6 cm and 10 cm, 4 cm apart. Its area is $\frac{1}{2}(6 + 10) \times 4 = 32$ cm^2.

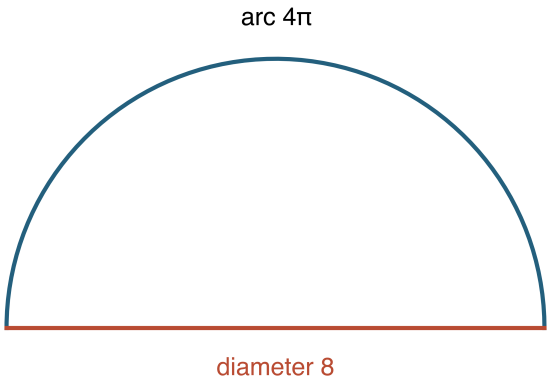
Trapezium area = average parallel sides × height



Area = ½(6 + 10) × 4 = 32 cm²

Worked example 7. A semicircle has diameter 8 cm. Its area is $\frac{1}{2} \times \pi \times 4^2 = 8\pi$ cm^2. Its perimeter is half the circumference plus the diameter: $4\pi + 8 = 20.6$ cm (3 s.f.). Leave answers "in terms of π" when asked.

Semicircle perimeter: curved arc + straight diameter



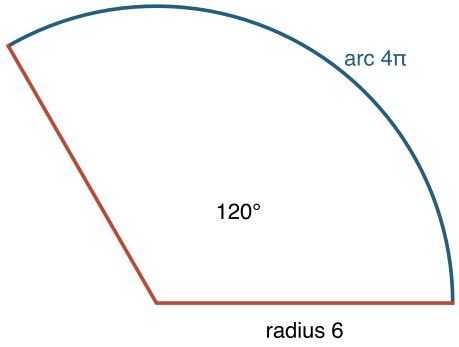
Perimeter = curved arc + diameter = 4π + 8

Arcs and sectors. A sector with angle θ is the fraction $\frac{\theta}{360}$ of the circle:

$$\text{arc length} = \frac{\theta}{360} \times 2\pi r \qquad \text{sector area} = \frac{\theta}{360} \times \pi r^2$$

Worked example 8. For $r = 6$ and $\theta = 120°$, the fraction is $\frac{1}{3}$: arc $= \frac{1}{3} \times 12\pi = 4\pi$ and area $= \frac{1}{3} \times 36\pi = 12\pi$.

One third of a turn: sector area 12π

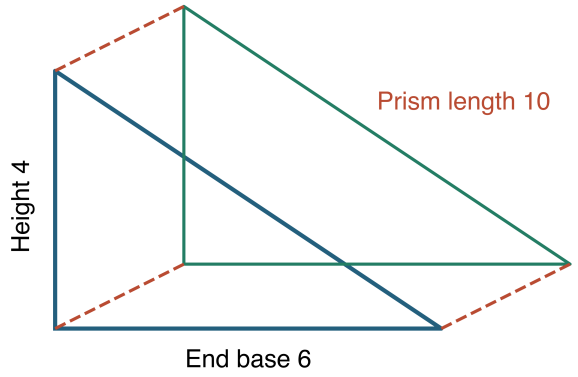


120° is one third of a turn: arc = $\frac{1}{3}$ × 12π = 4π, area = $\frac{1}{3}$ × 36π = 12π

Prisms and cylinders. The volume of a prism is the area of its cross-section times its length; a cylinder is a prism with a circular cross-section, $V = \pi r^2 h$.

Worked example 9. A triangular prism has an end triangle with base 6 cm and height 4 cm, and length 10 cm. The end area is $\frac{1}{2} \times 6 \times 4 = 12$ cm^2, so the volume is $12 \times 10 = 120$ cm^3.

Constant triangular end area 12 → volume 120



End area ½ × 6 × 4 = 12, volume 12 × 10 = 120

Pyramids, cones and spheres. A pyramid or cone has volume $\frac{1}{3}\times$ base area $\times$ height. A cone's curved surface area is πrl, where l is the slant height. A sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.

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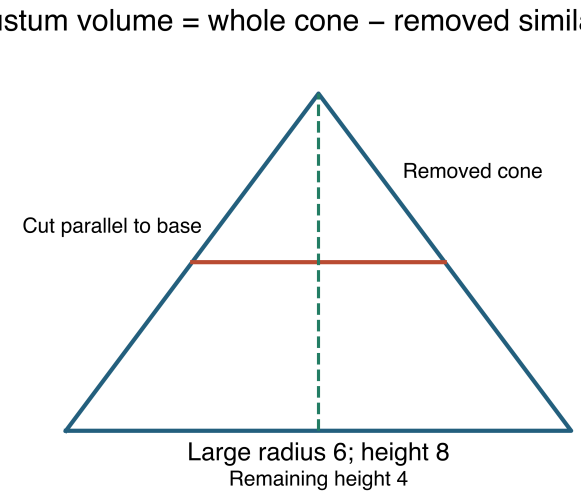


The pyramids of Giza: square-based pyramids, with volume one third of base area times height

Worked example 10. A cone has radius 3 cm and height 4 cm. Its slant height is $\sqrt{3^2+4^2} = 5$ cm. Volume $= \frac{1}{3}\pi \times 9 \times 4 = 12\pi$ cm^3; curved surface $= \pi \times 3 \times 5 = 15\pi$ cm^2; total surface, with the base, $= 15\pi + 9\pi = 24\pi$ cm^2.

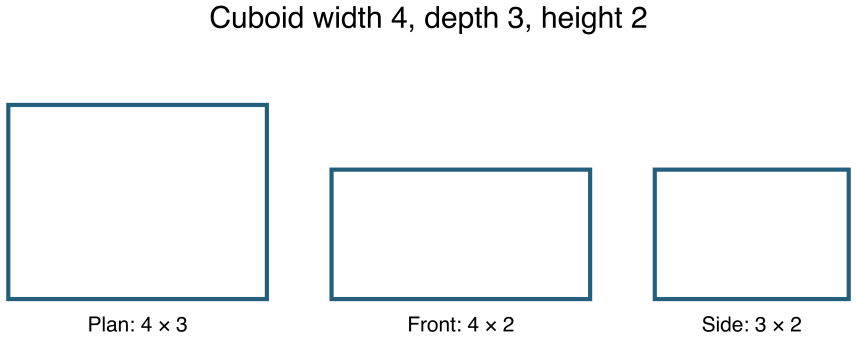
Worked example 11 (H). A **frustum** 截锥体 is a cone with its top cut off. Cut a cone of radius 6 cm and height 8 cm halfway up. The small cone on top has radius 3 cm and height 4 cm, so the frustum has volume $\frac{1}{3}\pi(36\times 8 - 9\times 4) = 96\pi - 12\pi = 84\pi$ cm^3. The small cone is similar with scale factor $\frac{1}{2}$, so its volume is $\frac{1}{8}$ of the whole cone.

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Frustum = whole cone − small cone; the small cone has half the height, so $\frac{1}{8}$ of the volume

Plans and elevations. A plan is the view from above; an **elevation** 立面图 is the view from the front or side. For a $4\times 3\times 2$ cuboid the plan is 4×3, the front elevation 4×2 and the side elevation 3×2.

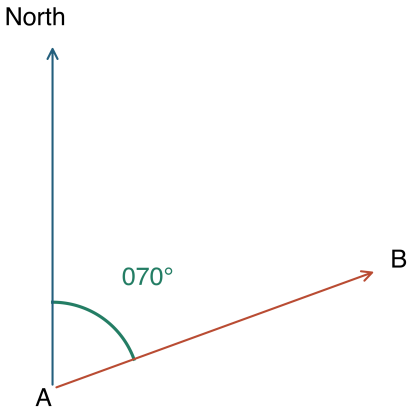


A 4 × 3 × 2 cuboid: each view shows two of its three dimensions

Bearings. A **bearing** 方位角 is measured clockwise from north and written with

three figures: east is 090°. If B is on a bearing of 070° from A, then A is on a bearing of $070° + 180° = 250°$ from B.

Clockwise from north at A; reverse bearing at B is 250°



Bearings run clockwise from north: 070° from A, so the back bearing from B is 070° + 180° = 250°

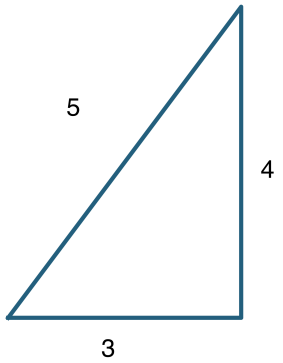
Pythagoras and trigonometry

Pythagoras' theorem. In a right-angled triangle, $a^2 + b^2 = c^2$, where c is the **hypotenuse** 斜边, the longest side, opposite the right angle. A ladder whose foot is 3 m from a wall and whose top is 4 m up the wall is $\sqrt{9+16} = 5$ m long.

Trigonometric ratios. Label the sides from the angle θ: opposite, **adjacent** 邻边 (next to the angle) and hypotenuse.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

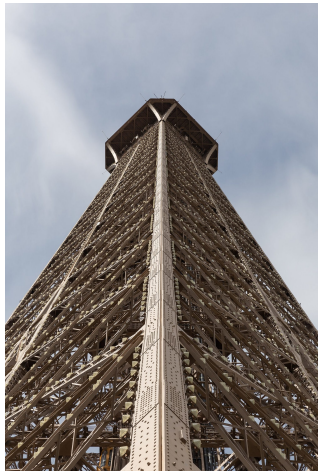
Label the known sides before choosing a rule



Label opposite, adjacent and hypotenuse first: 3, 4 and 5 here

Worked example 12. In a right-angled triangle the hypotenuse is 23.7 cm and x is the side adjacent to an angle of 52°. $\cos 52° = \frac{x}{23.7}$, so $x = 23.7\cos 52° = 14.6$ cm (1 d.p.).

Worked example 13. You stand 40 m from the foot of a tower and the angle of elevation of its top is 35°. The height is $40\tan 35° = 28.0$ m (3 s.f.).



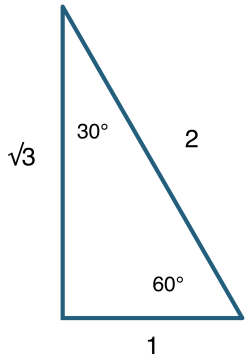
Looking up at a tower: the angle of elevation and a measured distance give its height with tan

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Exact values. Learn these, which come from half an equilateral triangle (sides 1, $\sqrt{3}$, 2) and a right isosceles triangle (sides 1, 1, $\sqrt{2}$). Each **exact value** 精确值 is written with fractions and surds.

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Bisected equilateral triangle: sides 1, √3, 2



sin 30° = 1/2, cos 30° = √3/2, tan 60° = √3 from the 1, √3, 2 triangle

Worked example 14. Show that $6\sin 30° + 2\cos 30° \times 4\tan 30°$ is a whole number. It equals $6 \times \frac{1}{2} + 2 \times \frac{\sqrt{3}}{2} \times \frac{4}{\sqrt{3}} = 3 + 4 = 7$.

Any triangle (H). For a triangle with sides a, b, c opposite angles A, B, C:

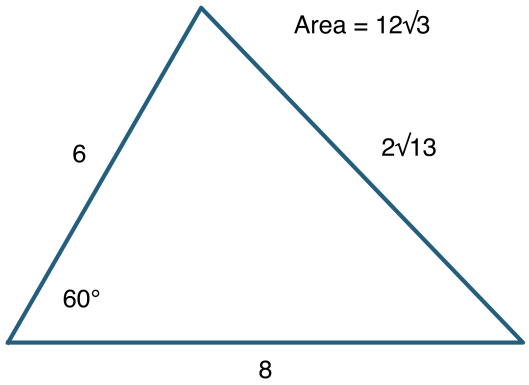
$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{area} = \frac{1}{2}ab\sin C$$

Use the cosine rule when you know two sides and the **included angle** 夹角 (the angle between them), or all three sides. Use the sine rule when you know a side and the angle opposite it.

Worked example 15 (H). Two sides of 6 cm and 8 cm meet at 60°. The third side: $c^2 = 36 + 64 - 2\times 6\times 8\times \frac{1}{2} = 52$, so $c = \sqrt{52} = 2\sqrt{13}$ cm. Area $= \frac{1}{2}\times 6\times 8\times \sin 60° = 12\sqrt{3}$ cm^2.

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Two sides and their included angle

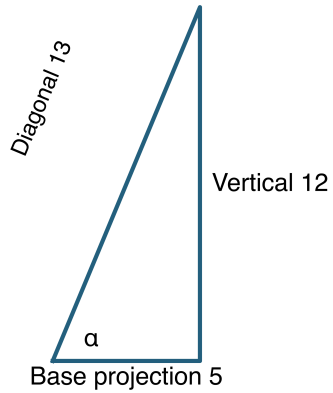


Area = ½ × 6 × 8 × sin 60° = 12√3; cosine rule gives the third side 2√13

Worked example 16 (H): two possible triangles. $a = 7$, $b = 10$ and $A = 40°$. The sine rule gives $\sin B = \frac{10\sin 40°}{7} = 0.918$, so $B = 66.7°$. But $180° - 66.7° = 113.3°$ has the same sine, and $40° + 113.3° < 180°$, so it is also possible. Two triangles fit the data. This can happen only when the side opposite the given angle is the shorter one.

3D problems (H). Find a right-angled triangle inside the solid. In a cuboid 3 by 4 by 12, the base diagonal is $\sqrt{3^2 + 4^2} = 5$, and the space diagonal is $\sqrt{5^2 + 12^2} = 13$. The angle α between the space diagonal and the base has $\tan\alpha = \frac{12}{5}$, so $\alpha = 67.4°$.

Cuboid section: first √(3²+4²)=5, then √(5²+12²)=13

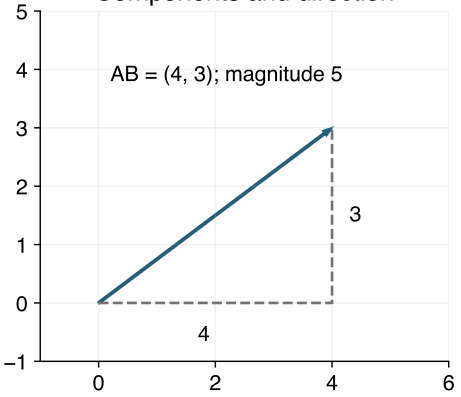


Base diagonal $\sqrt{(3^2 + 4^2)} = 5$, then space diagonal $\sqrt{(5^2 + 12^2)} = 13$; $\tan\alpha = 12/5$

Vectors and coordinate geometry

Vectors. A vector has size and direction. The column vector $\begin{pmatrix}4\\3\end{pmatrix}$ means 4 right and 3 up; its length is $\sqrt{4^2+3^2}=5$. Add and subtract vectors by adding and subtracting their parts: with $\mathbf{a}=\begin{pmatrix}4\\1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\3\end{pmatrix}$, $\mathbf{a}+\mathbf{b}=\begin{pmatrix}5\\4\end{pmatrix}$ and $2\mathbf{a}-\mathbf{b}=\begin{pmatrix}7\\-1\end{pmatrix}$.

Components and direction



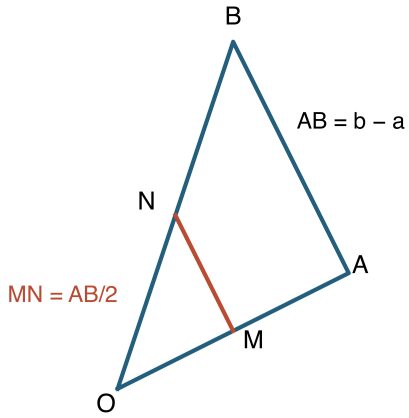
$AB = (4, 3)$ has magnitude $\sqrt{(4^2 + 3^2)} = 5$

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Vector proofs (H). If one vector is a number times another, the lines are parallel. Points are **collinear** 共线 (on one straight line) if two of the vectors joining them are parallel and share a point.

> **Worked example 17 (H).** In triangle OAB, $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. M and N are the midpoints of OA and OB. Then $\overrightarrow{MN}=\frac{1}{2}\mathbf{b}-\frac{1}{2}\mathbf{a}=\frac{1}{2}(\mathbf{b}-\mathbf{a})=\frac{1}{2}\overrightarrow{AB}$. So MN is parallel to AB and half as long.

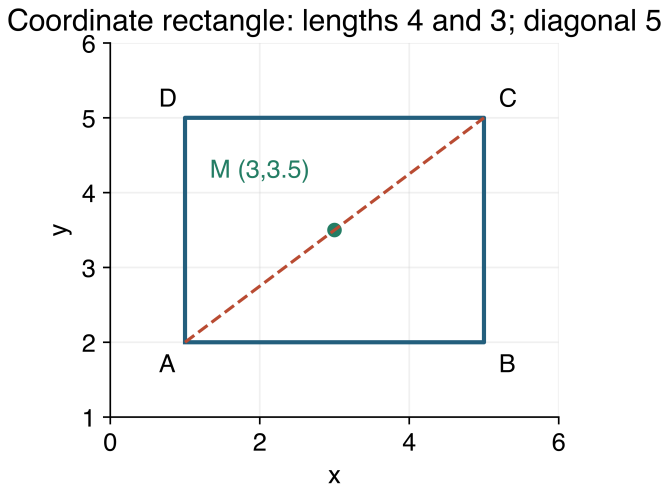
Midpoint segment: half length, parallel direction



$MN = ½b - ½a = ½(b - a) = ½AB$: half the length and parallel to AB

> **Worked example 18 (H).** $\overrightarrow{CD}=6\mathbf{a}+\mathbf{b}$, $\overrightarrow{CE}=14\mathbf{a}-3\mathbf{b}$ and $\overrightarrow{CF}=16\mathbf{a}-4\mathbf{b}$. Show that D, E and F are collinear. $\overrightarrow{DE}=\overrightarrow{CE}-\overrightarrow{CD}=8\mathbf{a}-4\mathbf{b}$ and $\overrightarrow{EF}=\overrightarrow{CF}-\overrightarrow{CE}=2\mathbf{a}-\mathbf{b}$. So $\overrightarrow{DE}=4\overrightarrow{EF}$: the lines are parallel and share the point E, so D, E and F lie on a straight line.

Coordinate geometry. Find horizontal and vertical lengths by subtracting coordinates, slanted lengths with Pythagoras, and midpoints by averaging. For $A(1,2)$, $B(5,2)$, $C(5,5)$ and $D(1,5)$: $AB=4$, $BC=3$, the diagonal $AC=5$ and its midpoint is $\left(\frac{1+5}{2},\frac{2+5}{2}\right)=(3,3.5)$.



Sides 4 and 3 give diagonal 5; the midpoint of AC is (3, 3.5)

Sources for this lesson

- AQA, *GCSE Mathematics (8300) specification*, subject content section 3.4 (Geometry and measures), and the GCSE formulae sheet.
- Cleomedes, *On the Circular Motions of the Celestial Bodies*, book 1, chapter 7, which records Eratosthenes' method.
- AQA, *GCSE Mathematics* Higher Papers 1 and 2 (June 2024), the trigonometry and vector questions, adapted.

Common errors

- Giving "Z-angles" or "F-angles" as reasons; name alternate or corresponding angles.
- Using the slant height instead of the perpendicular height in an area or volume formula.
- Forgetting the straight edge when finding the perimeter of a semicircle or sector.
- Labelling the hypotenuse as the adjacent side.
- Using the length scale factor for an area or volume.
- Rubbing out construction arcs.
- Taking only the calculator's acute angle when the sine rule allows two.
- Writing a bearing with fewer than three figures, such as 70° for 070°.

Key terms

- **Regular polygon**: a polygon with all sides and all angles equal.
- **Perpendicular bisector**: the line that cuts a segment in half at right angles.
- **Locus**: the set of points that follow a given rule.
- **Congruent**: exactly the same shape and size.
- **Scale factor**: the number every length is multiplied by in an enlargement or between similar shapes.
- **Sector**: the part of a circle between two radii and an arc.
- **Hypotenuse**: the longest side of a right-angled triangle, opposite the right angle.
- **Bearing**: a direction measured clockwise from north, written with three figures.
- **Collinear**: lying on one straight line.

Check yourself

1. Find the sum of the interior angles of an octagon, and the interior angle of a regular octagon.
2. Find the area of a circle of diameter 10 cm, in terms of π.
3. A sector has radius 9 cm and angle 40°. Find its arc length in terms of π.
4. A right-angled triangle has shorter sides 5 cm and 12 cm. Find the hypotenuse, and the smaller angle.
5. A cylinder has radius 2 cm and height 7 cm. Find its volume to 3 s.f.
6. (H) In triangle ABC, $a = 9$, $A = 50°$ and $B = 70°$. Find b.
7. (H) $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$. Explain why P, Q and R are collinear.

Answers: 1. $6 \times 180° = 1080°$; $1080° \div 8 = 135°$. 2. 25π cm^2. 3. $\frac{40}{360} \times 18\pi = 2\pi$ cm. 4. 13 cm; $\tan^{-1}\frac{5}{12} = 22.6°$. 5. $\pi \times 4 \times 7 = 88.0$ cm^3. 6. $b = \frac{9 \sin 70°}{\sin 50°} = 11.0$. 7. $\overrightarrow{QR} = 3\overrightarrow{PQ}$, so the lines are parallel, and they share the point Q.

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Probability

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Why it matters. On 9 September 1990 the writer Marilyn vos Savant answered a reader's puzzle in *Parade* magazine. On a game show there are three doors: behind one is a car, and behind the other two are goats. You choose a door. The host, who knows where the car is, opens another door to show a goat, and offers you the chance to switch. Should you? She said yes: switching wins the car two times out of three. About 10 000 readers wrote in to say she was wrong, nearly 1000 of them with doctorates. She was right. Think of 300 games. Your first choice is the car in about 100 of them, and then switching loses. In the other 200 your first choice is a goat, the host must open the other goat, and switching wins. Probability rewards careful counting more than first feelings, and that is what this topic teaches.

In this unit you will:

- use the probability scale and the fact that probabilities of all possible outcomes add up to 1;
- estimate probabilities from experiments and calculate expected frequencies;
- list outcomes systematically with tables, sample spaces and Venn diagrams;
- use tree diagrams for independent and dependent events;
- (higher tier) calculate conditional probabilities from tables, trees and Venn diagrams.

AQA GCSE Mathematics (8300), Probability (P1 to P9). Probability is tested on all three papers at both tiers; content marked (H) is Higher tier only. No probability formula is on the formulae sheet.

The probability scale and outcomes

The probability of an event is a number from 0 to 1 on the **probability scale** 概率标度. 0 means impossible, 1 means certain, and 0.5 means an even chance. Write probabilities as fractions, decimals or percentages, never as "1 in 4" or "1 : 4".

When every **outcome** 结果 is **equally likely** 等可能, the **theoretical probability** 理论概率 is

$$P(\text{event}) = \frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}$$

For a fair six-sided die, $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$.



Dice: roll many times and the relative frequency of each score settles towards its probability

Outcomes are **mutually exclusive** 互斥 if they cannot happen together, and a list of outcomes is **exhaustive** 穷尽 if one of them must happen. The probabilities of an exhaustive list of mutually exclusive outcomes add up to 1. So the probability that an event does not happen is $P(\text{not } A) = 1 - P(A)$.

> **Worked example 1.** A spinner lands on red, blue or green. $P(\text{red}) = 0.3$ and $P(\text{blue}) = 0.45$. Then $P(\text{green}) = 1 - 0.3 - 0.45 = 0.25$. If someone claims probabilities of 0.4, 0.5 and 0.2 for the three colours, they must be wrong: these add up to 1.1.

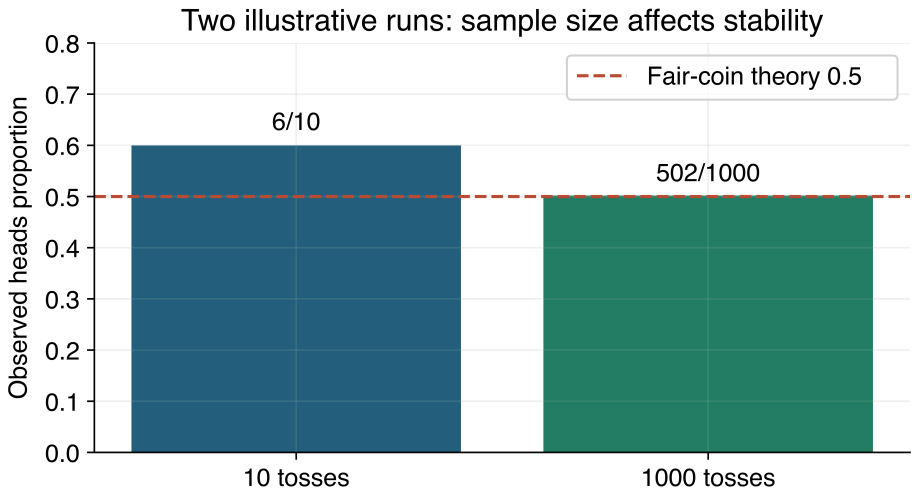
Experiments and expected frequency

When outcomes are not equally likely, such as a drawing pin landing point up, we estimate the probability by experiment. The **relative frequency** 相对频率 is

$$\text{relative frequency} = \frac{\text{number of times the event happened}}{\text{number of trials}}$$

A larger **sample size** 样本量 gives a better estimate. Six heads in 10 tosses gives 0.6, but 502 heads in 1000 tosses gives 0.502, much closer to the true 0.5 of a fair coin. A large sample settles near the true value, but it is never guaranteed to be exact. If four students flip the same coin different numbers of times, the best estimate comes from the student who flipped it most.

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A small run can be far from 0.5; a large run settles near it: 0.6 against 0.502

Expected frequency. If the probability of an event is p, then in n trials you expect it about $n \times p$ times. This **expected frequency** 期望频数 is a long-run average, not a promise.

> **Worked example 2.** The probability of rain on a June day in a town is 0.38. How many dry days would you expect in 150 days? $P(\text{dry}) = 1 - 0.38 = 0.62$, so the expected number is $0.62 \times 150 = 93$ days.

A **frequency tree** 频率树 shows the same information as numbers instead of probabilities.



A frequency tree with expected counts: 57 rainy and 93 dry days out of 150

Fair or biased? A fair dice or coin has equally likely outcomes; an **unbiased** 无偏 sample is one chosen so that every member has the same chance. **Randomness** 随机性 means the result of one trial cannot be predicted, and **fairness** 公平性 means every outcome has its proper chance. **Bias** 偏差 is a lean towards some outcomes.

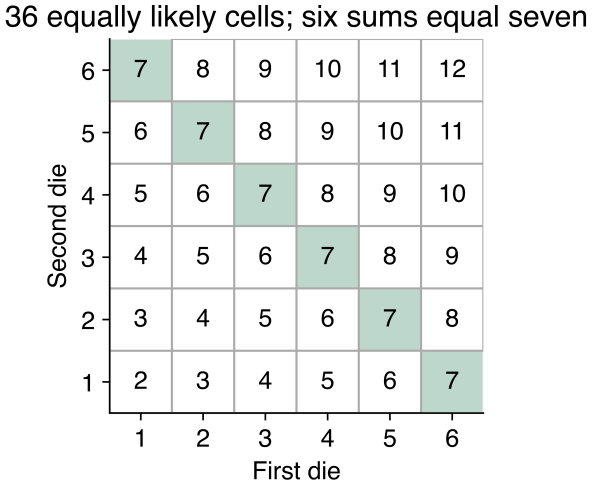
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> **Worked example 3.** Two dice are thrown 72 times, and a total of 7 comes up 14 times. In theory $P(7) = \frac{6}{36} = \frac{1}{6}$, so we expect $72 \times \frac{1}{6} = 12$ sevens. The relative frequency $\frac{14}{72}$ is a little higher than $\frac{1}{6}$, but small differences like this happen by chance all the time. It is not evidence of bias; many more throws would be needed.

Listing outcomes: sample spaces and Venn diagrams

Sample spaces. A **possibility space** 可能性空间, or sample space, lists every outcome of a combined experiment. Two dice give $6 \times 6 = 36$ equally likely outcomes.

> **Worked example 4.** Two fair dice are thrown and the scores added. Six of the 36 outcomes give 7: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2) and (6, 1). So $P(7) = \frac{6}{36} = \frac{1}{6}$. Three outcomes give 11 or more, so $P(\text{at least 11}) = \frac{3}{36} = \frac{1}{12}$.

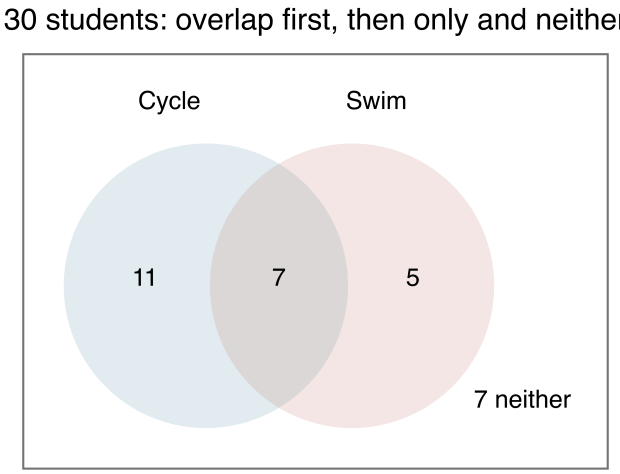


36 equally likely outcomes; six sum to 7, so P(7) = 6/36 = 1/6

Counting. Worked example 5. A code uses four different digits chosen from 1, 2, 3, 4 and 5. There are 5 choices for the first digit, 4 for the second, 3 for the third and 2 for the fourth: $5 \times 4 \times 3 \times 2 = 120$ codes. ($5^4 = 625$ would allow repeated digits.)

Venn diagrams. A **Venn diagram** 韦恩图 shows how groups overlap. The **intersection** 交集 $A \cap B$ is the overlap, "both A and B". The **union** 并集 $A \cup B$ is everything in A or B or both, "at least one". A' is everything not in A. Outside both circles are the members in **neither** 都不 group.

Worked example 6. In a class of 30, 18 students cycle, 12 swim and 7 do both. Fill the overlap first: 7. Then cycle only $= 18-7 = 11$ and swim only $= 12-7 = 5$. At least one: $11+7+5 = 23$, so neither $= 30-23 = 7$. $P(\text{cycle but not swim}) = \frac{11}{30}$.



Fill the overlap first: 11 + 7 + 5 = 23 do at least one, so 7 do neither

Two-way tables. A **two-way table** 双向表 shows the same counts in rows and columns. For this class: of the 18 cyclists, 7 swim and 11 do not; of the 12 non-cyclists, 5 swim and 7 do not. The column totals are 12 swimmers and 18 non-swimmers, and the grand total is 30.

Combined events and tree diagrams

Two events are **independent** 独立 if one does not change the probability of the other, like two separate coin tosses. For independent events, multiply:

$$P(A \text{ and } B) = P(A) \times P(B)$$

A **tree diagram** 树状图 shows each stage as a set of branches. Multiply along a path to find the probability of that path; add the results of different paths when more than one path gives the event.

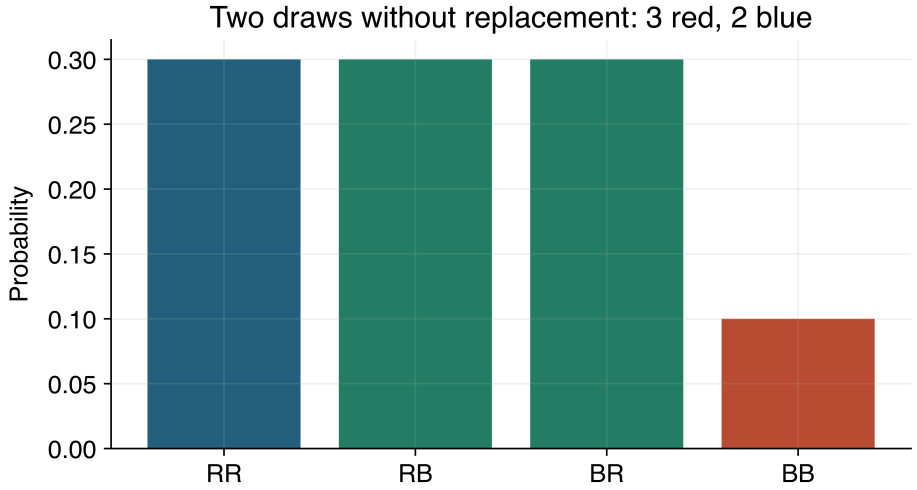
With and without replacement. A bag holds 3 red and 2 blue counters, and two are taken out. **With replacement** 放回, the first counter goes back, so the second draw is the same as the first: $P(RR) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$. **Without replacement** 不放回, the first counter stays out, so the second draw depends on the first: the events are dependent.

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Worked example 7. Without replacement:

- $P(RR) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$;
- $P(BB) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$;
- $P(\text{one of each}) = \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4} = \frac{12}{20} = \frac{3}{5}$.

Check: $\frac{3}{10} + \frac{1}{10} + \frac{3}{5} = 1$.



Without replacement: P(RR) = 3/5 × 2/4 = 0.3, P(BB) = 2/5 × 1/4 = 0.1

Worked example 8: "at least one". A machine breaks down on any day with probability 0.05, independently of other days. The probability of at least one breakdown in two days is $1 - P(\text{no breakdowns}) = 1 - 0.95 \times 0.95 = 1 - 0.9025 = 0.0975$. Using "1 minus none" is quicker than adding three paths.

Conditional probability (H)

A **conditional probability** 条件概率 is the probability of an event when you already know that another event has happened. $P(A \mid B)$ means "the probability of A, given B". Knowing B shrinks the group you count from to just the members of B:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

Worked example 9 (H). In the class of 30 (18 cycle, 12 swim, 7 both), $P(\text{swims} \mid \text{cycles}) = \frac{7}{18}$, because only the 18 cyclists count. But $P(\text{cycles} \mid \text{swims}) = \frac{7}{12}$: changing the condition changes the group.

Worked example 10 (H): a medical test. A condition affects 20% of 1000 people. A test is positive for 90% of people with the condition and for 10% of people without it. Use expected frequencies: 200 people have the condition, and 180 of them test positive; 800 do not, and 80 of them test positive. So 260 people test positive, and $P(\text{condition} \mid \text{positive}) = \frac{180}{260} = 0.692$, not 0.9.

Given positive: use 180 out of 260, not 180 out of 200

	Positive	Negative	Total
Condition	180	20	200
No condition	80	720	800
Total	260	740	1000

Given a positive test, use the positive column: 180 out of 260, not 180 out of 200

Worked example 11 (H): reasoning backwards. From the bag of 3 red and 2 blue, two counters are taken without replacement. Given that the second counter is red, what is the probability that the first was blue? $P(\text{second red}) = \frac{3}{5}\times\frac{2}{4}+\frac{2}{5}\times\frac{3}{4} = \frac{12}{20}$, and $P(\text{blue then red}) = \frac{6}{20}$. So the answer is $\frac{6}{20} \div \frac{12}{20} = \frac{1}{2}$.

Sources for this lesson

- AQA, *GCSE Mathematics (8300) specification*, subject content section 3.5 (Probability).
- M. vos Savant, "Ask Marilyn", *Parade* (9 September 1990), and her follow-up columns of December 1990 and February 1991.
- AQA, *GCSE Mathematics* Higher Papers 2 and 3 (June 2024), the expected-frequency, dice and machine questions, adapted.

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Common errors

- Writing a probability as "1 in 4" or as a ratio.
- Forgetting that the probabilities of all outcomes add up to 1.
- Adding along a branch of a tree diagram instead of multiplying.
- Keeping the same denominator for the second draw without replacement.
- Treating an expected frequency as certain.
- Counting the overlap twice in a Venn diagram.
- (H) Using the whole group as the denominator of a conditional probability.

Key terms

- **Outcome**: one possible result of an experiment.
- **Mutually exclusive**: events that cannot happen at the same time.
- **Relative frequency**: the number of times an event happened divided by the number of trials.
- **Expected frequency**: the number of times an event should happen on average: trials × probability.
- **Independent**: events where one does not change the probability of the other.
- **Sample space**: a list or table of all possible outcomes.
- **Conditional probability**: the probability of an event given that another event has happened.

Check yourself

1. A bag has 4 red, 5 green and 1 yellow counter. Find $P(\text{green})$ and $P(\text{not red})$.
2. A dice is thrown 300 times. How many sixes would you expect if it is fair?
3. A coin is tossed and a dice is thrown. How many outcomes are there, and what is $P(\text{head and 6})$?
4. $P(A) = 0.6$ and $P(B) = 0.3$ for independent events. Find $P(A \text{ and } B)$.
5. Two counters are taken without replacement from a bag of 4 red and 6 blue. Find $P(\text{both blue})$.
6. In a group of 40, 25 like tea, 20 like coffee and 8 like neither. How many like both?
7. (H) In question 6, find $P(\text{likes coffee} \mid \text{likes tea})$.

Answers: 1. $\frac{5}{10} = \frac{1}{2}$; $\frac{6}{10} = \frac{3}{5}$. 2. $300 \times \frac{1}{6} = 50$. 3. 12 outcomes; $\frac{1}{12}$. 4. $0.6 \times 0.3 = 0.18$. 5. $\frac{6}{10} \times \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$. 6. $40 - 8 = 32$ like at least one, so $25 + 20 - 32 = 13$ like both. 7. $\frac{13}{25}$.

Statistics

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Why it matters. In 1936 the American magazine *The Literary Digest* ran the biggest opinion poll the world had seen. It posted about 10 million voting cards, got about 2.4 million back, and predicted that Alf Landon would win the presidential election easily. Franklin Roosevelt won 46 of the 48 states. A young pollster, George Gallup, predicted the result correctly with a far smaller sample. The magazine's mistake was in who it asked: its addresses came largely from telephone directories and car registrations, and in the middle of the Great Depression people with phones and cars were richer than average. In addition, only a quarter of the people asked replied. A huge sample cannot fix a biased one. Statistics is about collecting data fairly, showing it clearly and describing it honestly.

In this unit you will:

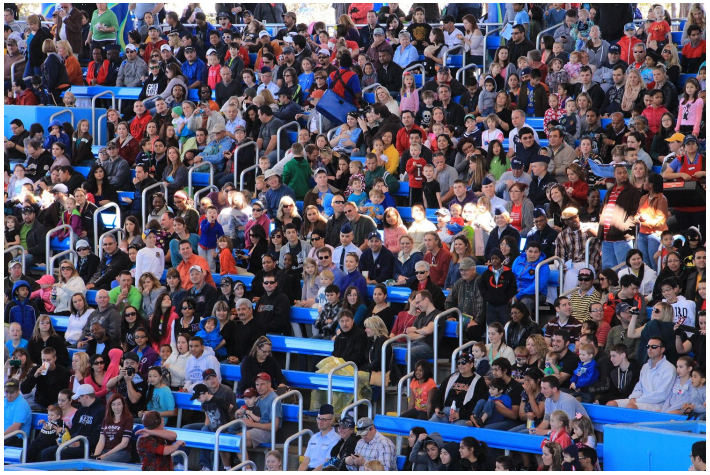
- understand samples and populations, and the limits of what a sample can tell you;
- draw and read tables, bar charts, pie charts and time-series graphs;
- calculate and compare averages and measures of spread, including from frequency tables;
- (higher tier) draw and read histograms, cumulative frequency graphs and box plots;
- use scatter graphs, correlation and lines of best fit.

AQA GCSE Mathematics (8300), Statistics (S1 to S6). Statistics is tested on all three papers at both tiers; content marked (H) is Higher tier only. No statistics formula is on the formulae sheet.

Samples and data

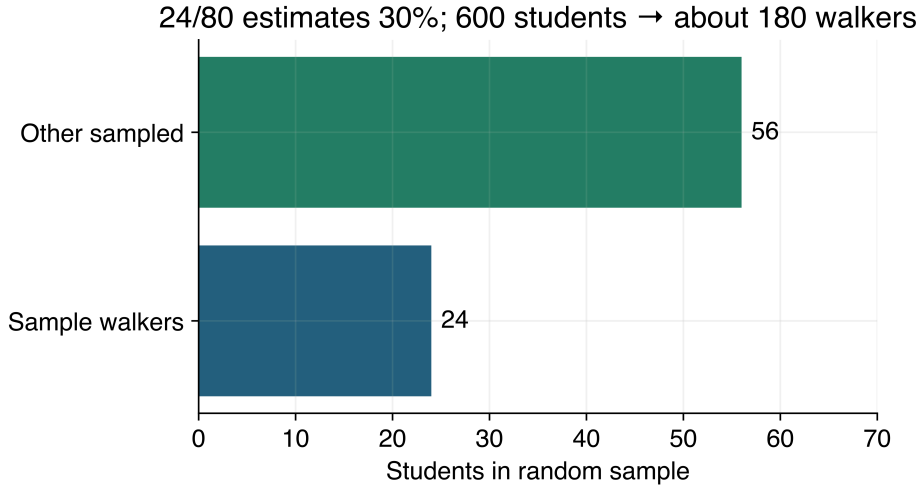
The **population** 总体 is the whole group you want to know about. A **census** 普查 collects data from every member; a **sample** 样本 collects data from only some of them. A sample is cheaper and quicker, but it can mislead. A random sample, where every member has the same chance of being chosen, avoids choosing people unfairly, but even a random sample varies by chance.

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A crowd is a population: a sample must be chosen fairly to tell us about all of it

Worked example 1. In a random sample of 80 students at a school of 600, 24 walk to school. The proportion is $\frac{24}{80} = 0.3$, so about $0.3 \times 600 = 180$ students in the school walk. This is an estimate, not a count.



24 out of 80 is 30%, so about 0.3 × 600 = 180 walkers in the school

Biased samples. Asking only members of a sports club would give too many active students. An online poll that people choose to answer hears only from those who care. Making such a sample larger does not remove the bias.

Types of data.

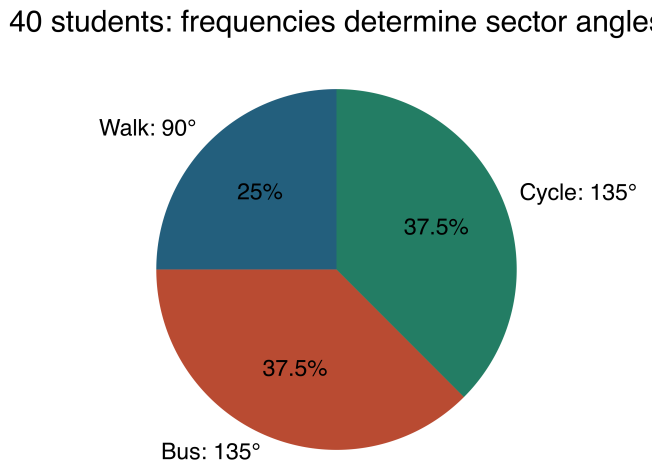
- **Primary data** 原始数据 is collected by you; **secondary data** 二手数据 was collected by someone else.
- **Categorical data** 分类数据 is in words, such as favourite colour.
- **Discrete** 离散 data can take only certain values, such as the number of brothers and sisters.
- **Continuous data** 连续数据 can take any value in a range, such as a time or a height, even if it is rounded when it is recorded.

Tables, charts and time series

Choose a chart that suits the data. Bar charts, pictograms and **pie charts** 饼图 suit categorical data. Vertical line charts suit discrete numbers. A **time series** 时间序列 shows how something changes over time, such as monthly rainfall, and is drawn as a line graph with the points joined in time order.

Pie charts. The whole circle, 360°, stands for the whole total. Each item gets $\frac{\text{frequency}}{\text{total}} \times 360°$.

> **Worked example 2.** 40 students travel to school: 10 walk, 15 cycle and 15 take the bus. Each student is $360° \div 40 = 9°$, so walk $= 90°$, cycle $= 135°$ and bus $= 135°$. Going the other way, a 162° sector in a pie chart is $\frac{162}{360} = 45\%$ of the total.



Each student is 360° ÷ 40 = 9°: 10 walkers make 90°, 15 cyclists 135°

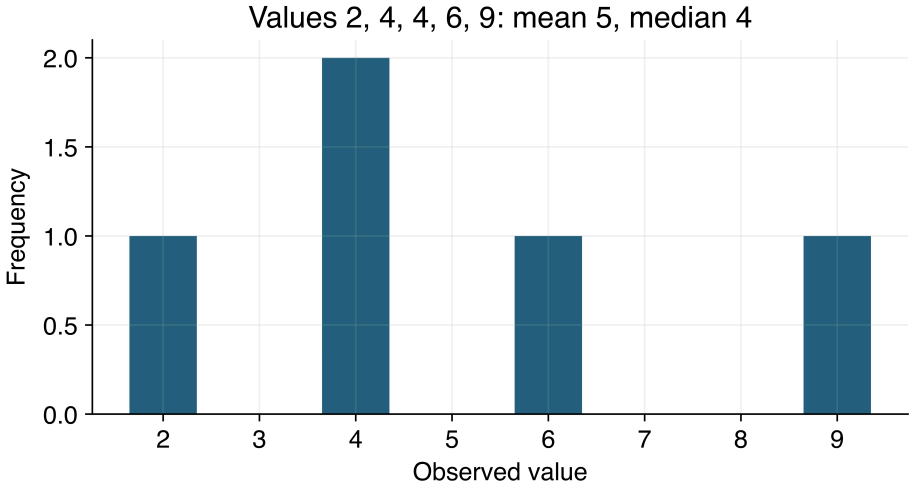
Averages and spread

The three averages. Each average is a measure of **central tendency** 集中趋势, a typical value.

- The **mean** 平均数 is the total divided by the number of values.
- The **median** 中位数 is the middle value when the data is in order.
- The **mode** 众数 is the most common value; for grouped data, the **modal class** 众数组 is the group with the highest frequency.

Spread. The **range** 极差 is the largest value minus the smallest. It is easily distorted by one extreme value.

> **Worked example 3.** For 2, 4, 4, 6, 9: the mean is $25 \div 5 = 5$, the median is 4, the mode is 4 and the range is $9 - 2 = 7$. If the 9 were replaced by an **outlier** 异常值 of 24, the mean would rise to 8, but the median would stay at 4. The median is a better average when there are extreme values.



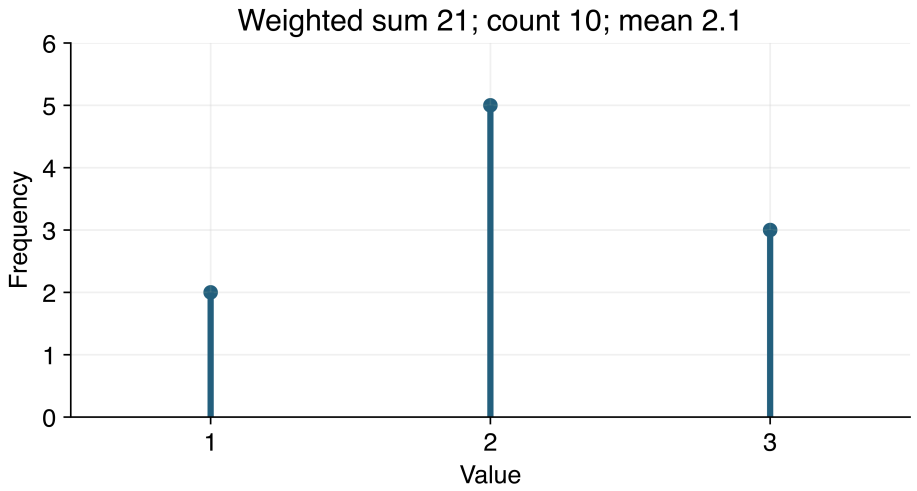
For 2, 4, 4, 6, 9: mean = 25 ÷ 5 = 5, median = 4

Means from frequency tables. Multiply each value by its frequency, add the results, and divide by the total frequency:

$$\text{mean} = \frac{\sum fx}{\sum f}$$

> **Worked example 4.** The values 1, 2 and 3 have frequencies 2, 5 and 3. The mean is $\frac{1\times2+2\times5+3\times3}{10} = \frac{21}{10} = 2.1$.

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Mean = (1×2 + 2×5 + 3×3) ÷ 10 = 21 ÷ 10 = 2.1

Worked example 5. Forty lengths are measured: 15 are 7.2 cm, 18 are 7.6 cm and 7 are 8.0 cm. $\sum fx = 108 + 136.8 + 56 = 300.8$, so the mean is $300.8 \div 40 = 7.52$ cm. For grouped data, use the midpoint of each class as x; the answer is then only an estimate, because you do not know the exact values.

Comparing two sets of data. Always compare an average and a spread, and say what they mean in the context. For example: "The swimming club's median age is lower, so its members are younger on average. Its interquartile range is higher, so their ages are less consistent."

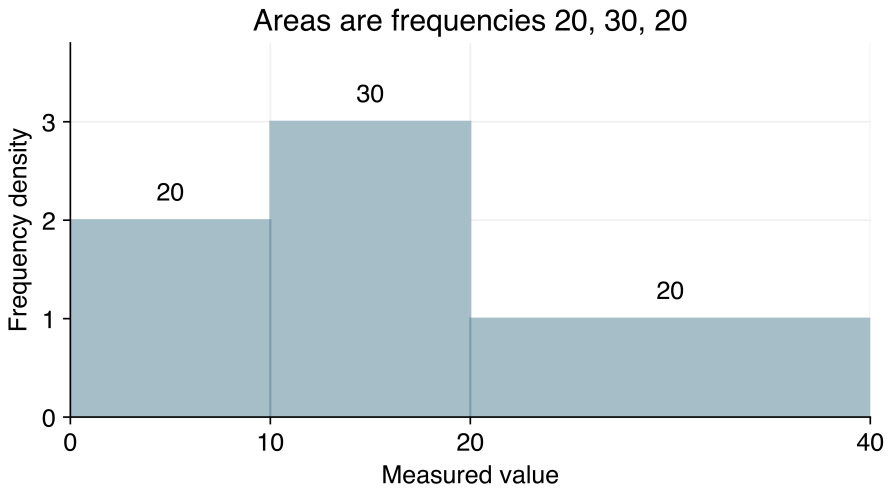
Histograms, cumulative frequency and box plots (H)

Histograms. In a **histogram** 直方图 the area of each bar, not its height, shows the frequency. When the **class intervals** 组距 have different widths, the height is the **frequency density** 频率密度:

$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}$$

Worked example 6 (H). Classes $0 \le x < 10$, $10 \le x < 20$ and $20 \le x < 40$ have frequencies 20, 30 and 20. The **class width** 组宽 of the last class is 20, so the frequency densities are $20 \div 10 = 2$, $30 \div 10 = 3$ and $20 \div 20 = 1$. The wide last bar is the shortest, but it still holds 20 values.

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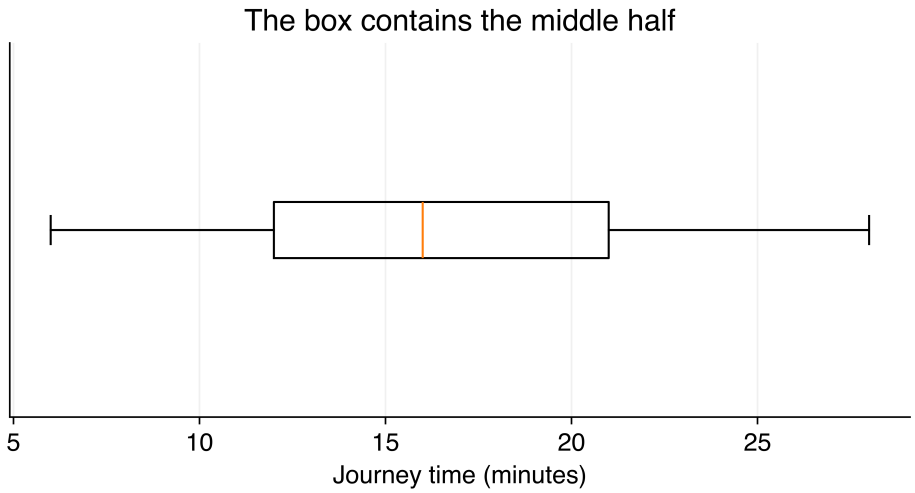
Area = frequency: 10 × 2 = 20, 10 × 3 = 30, 20 × 1 = 20

Cumulative frequency. The **cumulative frequency** 累积频率 is a running total. Plot each running total at the upper end of its class and join the points with a smooth curve or straight lines, starting at zero. For the classes above, the points are $(0, 0)$, $(10, 20)$, $(20, 50)$ and $(40, 70)$.

From the graph you can estimate the median (at half the total) and the **quartiles** 四分位数: the lower quartile Q_1 at one quarter of the total and the upper quartile Q_3 at three quarters. The **interquartile range** 四分位距 $Q_3 - Q_1$ is the spread of the middle half of the data, and outliers do not affect it.

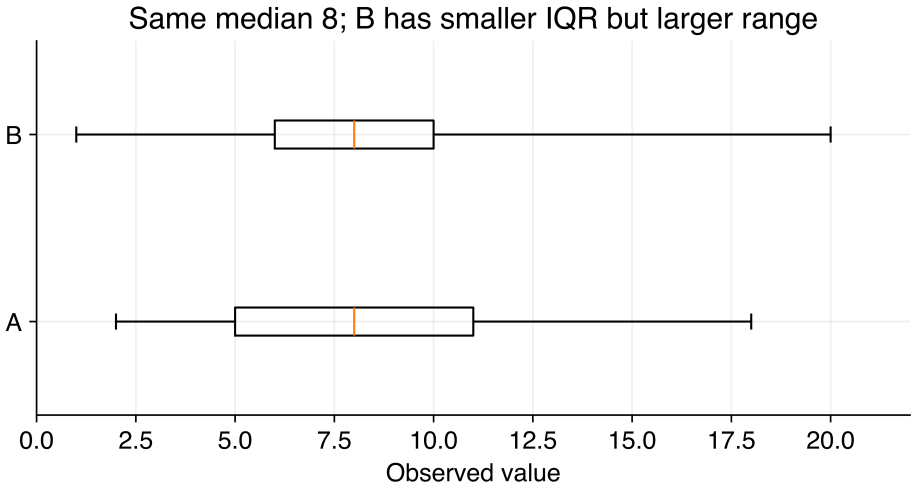
Worked example 7 (H). With 70 values: the median is at 35, which is about 15; Q_1 is at 17.5, about 8.75; Q_3 is at 52.5, about 22.5. So IQR $= 22.5 - 8.75 = 13.75$. To find how many values are above 25, read up from 25 to a cumulative frequency of about 55: so about $70 - 55 = 15$ values.

Box plots. A **box plot** 箱线图 shows five values: the minimum, Q_1, the median, Q_3 and the maximum. The box holds the middle half of the data.



The box holds the middle half: quartiles 12 and 21 minutes, IQR 9, median 16

Worked example 8 (H). Two box plots, A and B, have the same median, 8. B has the smaller IQR (4 against 6), so its middle half is more consistent, but B has the larger range, so its extreme values are further apart.



Same median 8; B has the smaller IQR (4 against 6) but the larger range

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Scatter graphs and correlation

A **scatter graph** 散点图 plots pairs of values, such as hours of study and test score, as points. Do not join the points.

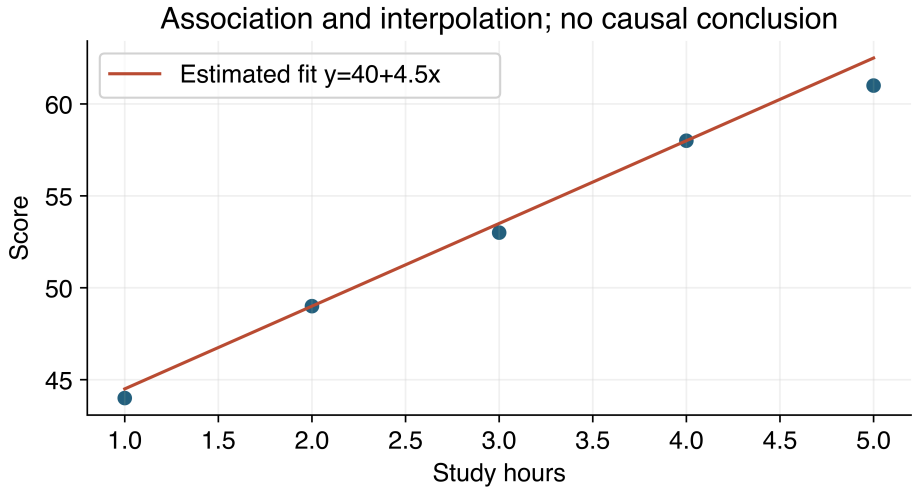
Correlation 相关性 describes the pattern:

- positive correlation: as one value increases, the other tends to increase;
- negative correlation: as one value increases, the other tends to decrease;
- no correlation: no clear pattern.

Correlation can be strong (points close to a line) or weak. Correlation does not prove that one thing causes the other: ice cream sales and sunburn cases rise together because both depend on sunny weather.

Lines of best fit. Draw a **line of best fit** 最佳拟合线 by eye through the middle of the points, with about as many points on each side. It does not have to pass through the origin.

Worked example 9. For study hours from 1 to 5 and scores from 44 to 61, the line of best fit is $y = 40 + 4.5x$. For 3.6 hours it predicts $40 + 4.5 \times 3.6 = 56.2$ marks. This is safe because 3.6 is inside the data: we **interpolate** 内插. For 10 hours the line predicts 85 marks, but 10 is far outside the data, and scores may stop rising: to **extrapolate** 外推 like this is risky.



Use the line to interpolate inside the data, for example 4.5 → 60; it shows association, not cause

Sources for this lesson

- AQA, *GCSE Mathematics (8300) specification*, subject content section 3.6 (Statistics).
- "Landon, 1,293,669; Roosevelt, 972,897", *The Literary Digest* (31 October 1936).
- P. Squire, "Why the 1936 Literary Digest poll failed", *Public Opinion Quarterly* 52 (1988), pages 125 to 133.
- AQA, *GCSE Mathematics* Higher Papers 1, 2 and 3 (June 2024), the pie chart, frequency table and box plot questions, adapted.

Common errors

- Thinking a larger sample removes bias.
- Using the class width instead of the midpoint to estimate a mean.
- Dividing $\sum fx$ by the number of classes instead of the total frequency.
- (H) Drawing histogram bars with height equal to frequency when the class widths differ.
- (H) Plotting cumulative frequency at the midpoint instead of the upper end of the class.
- Comparing only averages; a comparison also needs a measure of spread.
- Saying that correlation proves cause.

Key terms

- **Population**: the whole group being studied.
- **Sample**: part of a population, used to learn about the whole.
- **Mean**: the total of the values divided by the number of values.
- **Median**: the middle value when the data is in order.
- **Range**: the largest value minus the smallest.
- **Interquartile range**: the upper quartile minus the lower quartile.
- **Frequency density**: frequency divided by class width.
- **Correlation**: a relationship between two variables shown on a scatter graph.

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Check yourself

1. Find the mean, median, mode and range of 3, 7, 7, 8, 10.
2. A pie chart shows 72 people. How many degrees stand for each person, and what angle stands for 20 people?
3. Values 0, 1, 2, 3 have frequencies 4, 6, 7, 3. Find the mean.
4. A random sample of 50 trees in a forest of 2000 has 12 oak trees. Estimate the number of oak trees in the forest.
5. (H) A class $10 \le t < 25$ has frequency 45. Find its frequency density.
6. (H) A box plot has $Q_1 = 14$, median 19 and $Q_3 = 26$. Find the interquartile range.
7. Describe the correlation between outdoor temperature and heating bills, and say whether one causes the other.

Answers: 1. Mean $35 \div 5 = 7$, median 7, mode 7, range 7. 2. $360 \div 72 = 5°$ each; $100°$. 3. $\frac{0+6+14+9}{20} = \frac{29}{20} = 1.45$. 4. $\frac{12}{50} \times 2000 = 480$. 5. $45 \div 15 = 3$. 6. $26 - 14 = 12$. 7. Negative correlation; here colder weather does cause more heating, but the graph alone cannot show that.