

# Algebra – native teaching supplement

AQA · GCSE · Mathematics

## Supported teaching and tier boundary

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8300: Algebra. Version: Version 1.0, 12 September 2014; first examination 2017.

Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

### ■ Equations, identities and rearrangement · Foundation

An equation asks which inputs satisfy an equality; an identity holds for all allowed inputs. Preserve equality by applying the same operation to both sides. State restrictions before dividing by a variable.

$$C_1 = 20 + 3x, \quad C_2 = 44 + x, \quad C_1 = C_2$$

$20+3x=44+x$  gives  $2x=24$  and  $x=12$ . Both plans then cost 56. In  $A=\pi r^2$ , divide by  $\pi$  and take the positive square root to obtain  $r=\sqrt{(A/\pi)}$ , because  $r$  is a length.

Cancelling a term is not the same as cancelling a factor. In  $(x^2+2x)/x$ , factor the numerator and retain  $x \neq 0$ . Check a rearrangement by substitution.

Define the unknown and set up a linear equation. Check the answer by substitution. Restrict this Foundation/Core lesson to simple expressions and equations.

### ■ Equations, identities and rearrangement · Higher

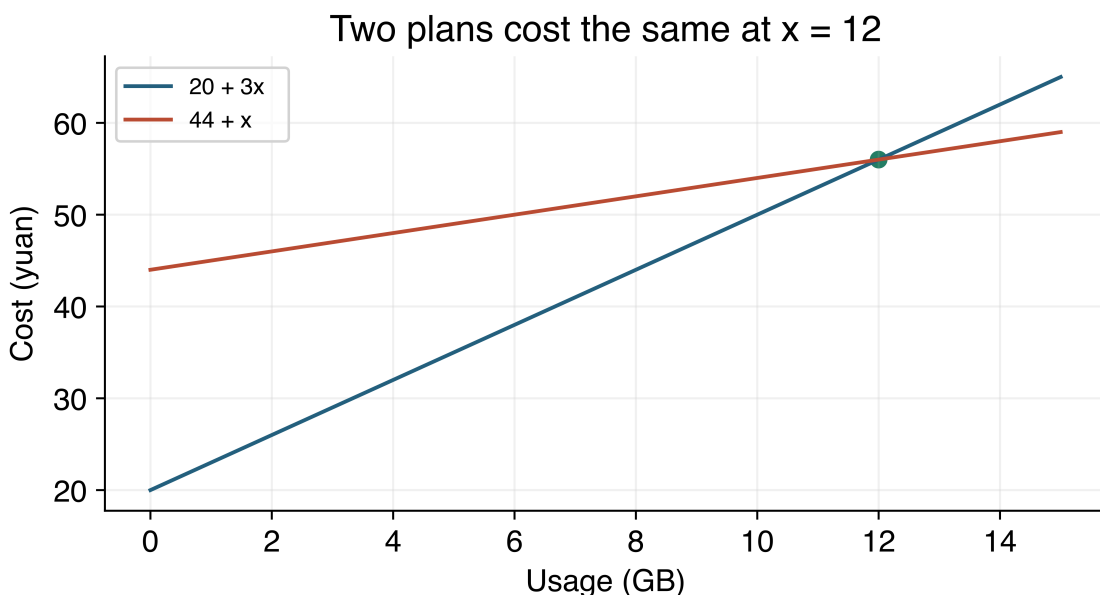
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Cancelling a term is not the same as cancelling a factor. In  $(x^2+2x)/x$ , factor the numerator and retain  $x \neq 0$ . Check a rearrangement by substitution.

Set up the equation from units and the meaning of the unknown. A negative or fractional solution may be algebraically correct but impossible for a count.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Factorising and solving simple quadratics · Foundation

Expand brackets and factorise simple quadratics. Solve by setting each factor equal to zero, and use a graph to interpret the roots.

$$(x - 3)(x - 7) = 0$$

$x^2 - 10x + 21 = (x - 3)(x - 7)$ . Hence the equation  $x^2 - 10x + 21 = 0$  has roots 3 and 7. Check each root by substitution and mark both intercepts on the graph.

Multiplying an inequality by a negative number reverses its direction. A sketch must show which side of each root satisfies the inequality. Geometry may restrict  $x$  further. This Foundation/Core lesson uses factorisation and graphical roots; the discriminant, quadratic formula and quadratic inequalities are reserved for the advanced tier.

### ■ Quadratics and inequalities · Higher

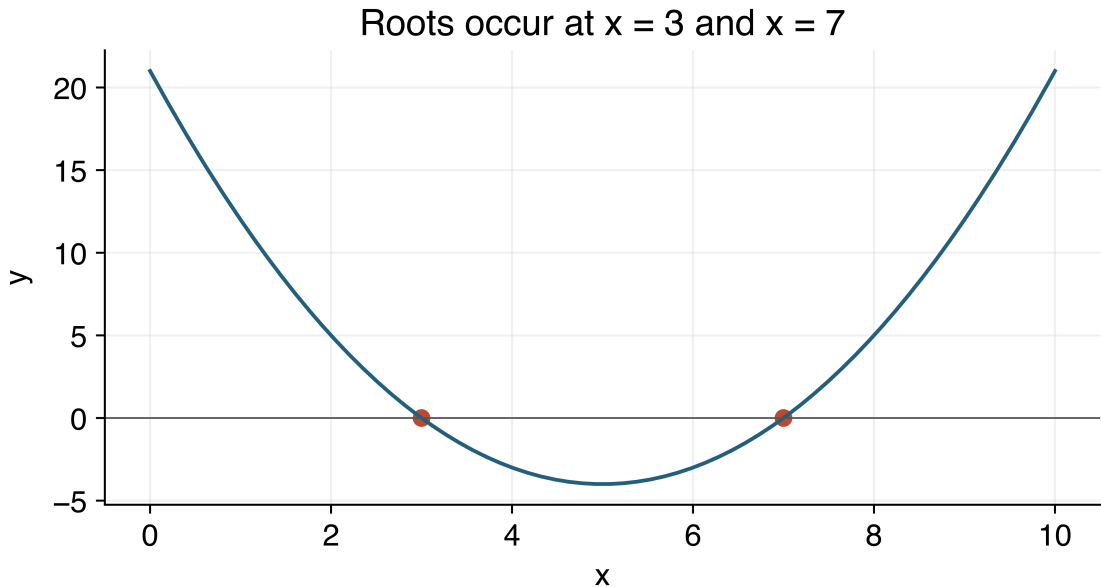
Factor where possible; otherwise complete the square or use the quadratic formula. A quadratic inequality needs the sign on intervals, not only the roots. The discriminant identifies repeated or missing real roots.

$$ax^2 + bx + c = 0, \quad \Delta = b^2 - 4ac$$

The equation  $x^2 - 10x + 21 = 0$  factorises as  $(x - 3)(x - 7) = 0$ , giving roots 3 and 7. Complete the square:  $x^2 - 10x + 21 = (x - 5)^2 - 4$ , giving turning point (5, -4) and symmetry line  $x = 5$ . The formula  $x = [-b \pm \sqrt{(b^2 - 4ac)}] / (2a)$  also gives  $(10 \pm 4) / 2 = 3, 7$ . For  $2x^2 + 5x - 3 = 0$ ,

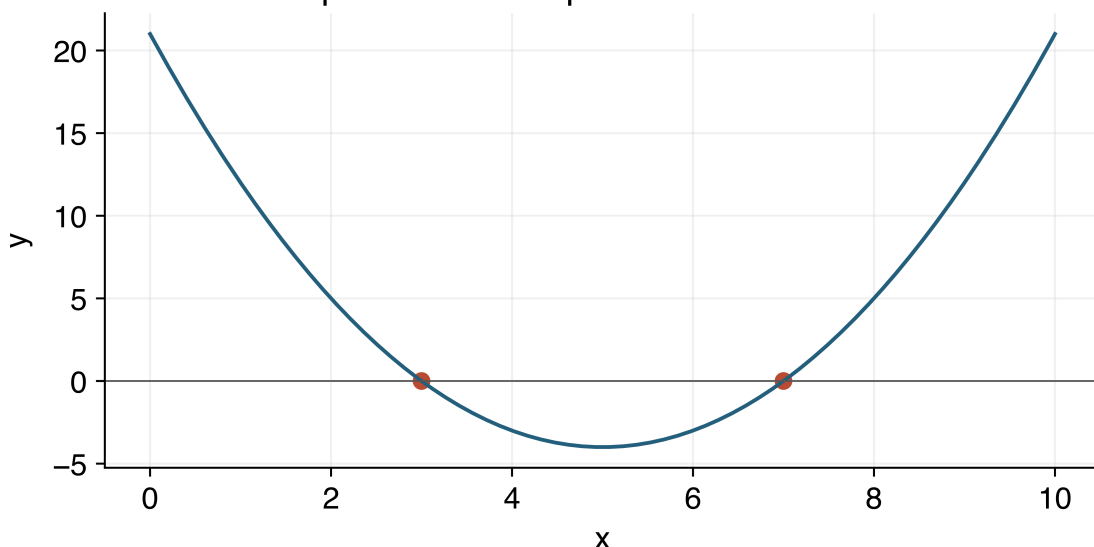
the discriminant is 49 and roots are  $(-5 \pm 7)/4 = 1/2, -3$ . A graph gives approximate roots when exact factorisation is inconvenient. For an enclosure with area  $A = x(10 - x)$ , complete the square to get  $A = 25 - (x - 5)^2$ . The greatest area is 25 at  $x = 5$ , within  $0 < x < 10$ .

Multiplying an inequality by a negative number reverses its direction. A sketch must show which side of each root satisfies the inequality. Geometry may restrict  $x$  further. AQA A11/A18 Higher interprets roots, intercepts and turning points, completes the square and uses the quadratic formula, including equations needing rearrangement. Factorisation and substitution check each result.



*Original native-lesson illustration; labels belong to its worked example.*

### The quadratic is nonpositive between 3 and 7



*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Two linear simultaneous equations · Foundation

Multiply equations to make a variable cancel, or substitute an expression from one equation into the other. Solve the remaining linear equation and recover the second variable. The intersection is a point satisfying both original equations.

$$x + y = 12, \quad 3x + 2y = 31$$

For  $x+y=12$  and  $3x+2y=31$ , subtract twice the first equation from the second:  $x=7$ . Then  $y=5$ . Check both  $7+5=12$  and  $3 \times 7 + 2 \times 5 = 31$ . The two straight-line graphs meet at  $(7,5)$ .

Check the ordered pair in both original equations. Multiplying an equation means multiplying every term, including the constant. Parallel distinct lines have no common solution.

AQA A19 Foundation solves two linear equations by elimination or substitution and interprets the graph intersection. Linear/quadratic systems belong to Higher.

#### ■ Linear and linear/quadratic simultaneous equations · Higher

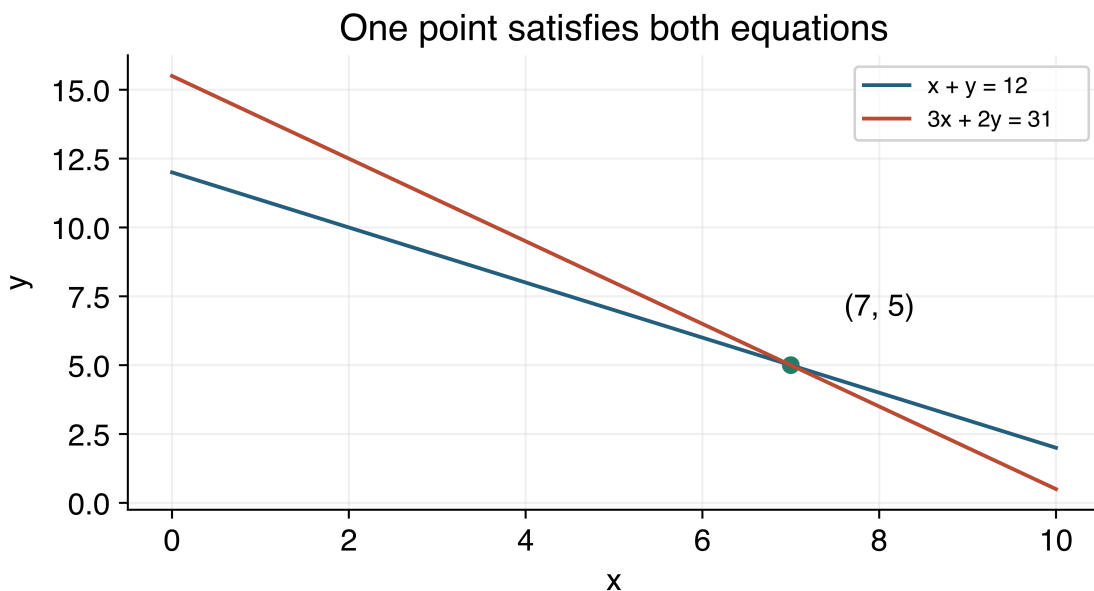
For two linear equations, use elimination or substitution and check both equations. For a line and a quadratic, substitute the linear relation first; then solve the resulting quadratic. For inequalities, shade the region satisfying every condition.

$$x + y = 12, \quad 3x + 2y = 31$$

For  $x+y=12$  and  $3x+2y=31$ , subtract twice the first equation to obtain  $x=7, y=5$ . For  $y=x+2$  and  $y=x^2$ , equate outputs:  $x^2-x-2=0$ , hence  $x=2$  or  $-1$ . The intersections are  $(2,4)$  and  $(-1,1)$ , and both satisfy the line and parabola.

One equation checked is not enough. A line can meet a quadratic twice, so retain both solutions unless the context removes one. Inequality boundaries may be included or excluded according to the sign.

AQA A19 Higher includes two linear equations and linear/quadratic systems. Elimination suits linear pairs; substitution reduces a line/curve pair to a quadratic. Retain every solution and check both original equations.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Straight lines and gradients · Foundation

Gradient is change in  $y$  divided by change in  $x$ . A straight line has  $y=mx+c$ , where  $c$  is its  $y$ -intercept. Parallel lines have equal gradients.

$$y = mx + c$$

Through  $(2,5)$  with gradient 3, substitute to get  $5=3 \times 2+c$ , so  $c=-1$  and  $y=3x-1$ . Points  $(1,2)$  and  $(4,8)$  give gradient  $(8-2)/(4-1)=2$ .

A vertical line has no finite gradient; do not force it into  $y=mx+c$ . Read the signs of a circle's centre carefully. The radius to a tangent is perpendicular to the tangent.

Plot a straight line using two checked points and label its intercept. Perpendicular-gradient formulae and circle equations are not part of this Foundation/Core lesson.

## ■ Coordinate geometry and tangents · Higher

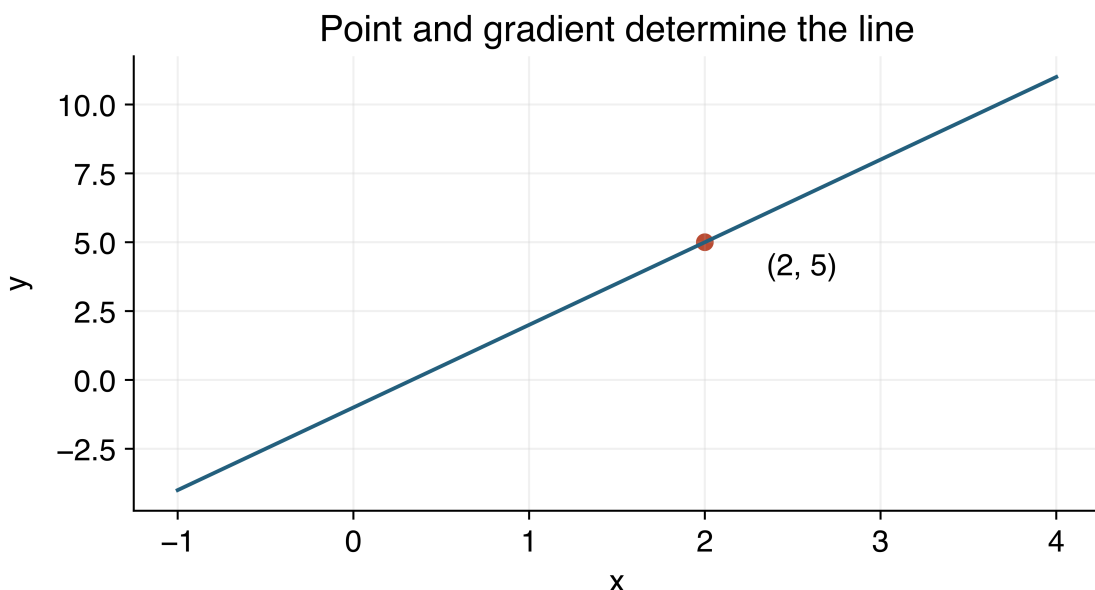
A line through  $(x_1, y_1)$  with gradient  $m$  has  $y - y_1 = m(x - x_1)$ . Parallel lines have equal gradients. Finite perpendicular gradients multiply to  $-1$ . A circle has  $(x - a)^2 + (y - b)^2 = r^2$ .

$$y - y_1 = m(x - x_1), \quad (x - a)^2 + (y - b)^2 = r^2$$

Through  $(2, 5)$  with gradient  $3$ ,  $y - 5 = 3(x - 2)$ , so  $y = 3x - 1$ . A perpendicular through the same point has  $y - 5 = -(x - 2)/3$ . The circle  $(x - 2)^2 + (y + 1)^2 = 25$  has centre  $(2, -1)$  and radius  $5$ .

A vertical line has no finite gradient; do not force it into  $y = mx + c$ . Read the signs of a circle's centre carefully. The radius to a tangent is perpendicular to the tangent.

Before solving a line-circle intersection, predict whether there are zero, one or two intersections. Substitution produces a quadratic whose discriminant checks the prediction.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Arithmetic sequences and $n$ th terms · Foundation

Find a constant difference for an arithmetic sequence. Its  $n$ th term is  $a + (n - 1)d$ . A term-to-term rule describes how to reach the next term; a position-to-term rule gives a term directly.

$$u_n = a + (n - 1)d$$

For 5,8,11,14,... the common difference is 3. The  $n$ th term is  $5+3(n-1)=3n+2$ . At  $n=8$ ,  $u_8=26$ . To find the position of 62, solve  $3n+2=62$ , giving  $n=20$ .

The first term has index 1, so the exponent is  $n-1$ . A sequence is a list; a series is a sum. A geometric sequence can alternate in sign and still converge.

Generate several terms and check a proposed  $n$ th-term rule. Infinite geometric series and advanced sum formulae are excluded from this Foundation/Core lesson.

## ■ Arithmetic and finite geometric sequences · Higher

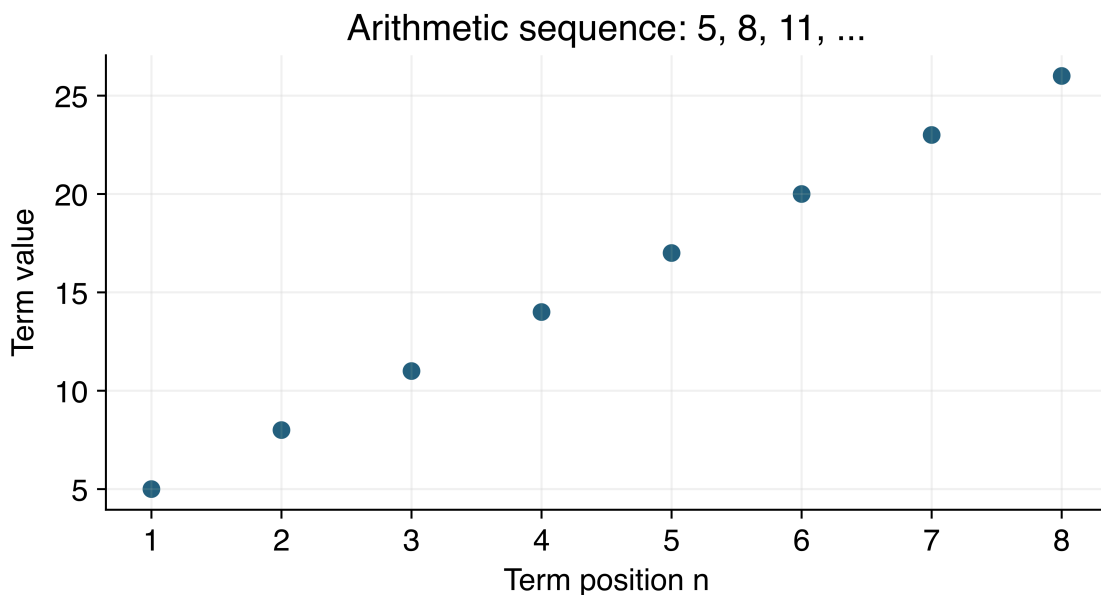
An arithmetic sequence adds a constant difference and has  $n$ th term  $a+(n-1)d$ . A geometric sequence multiplies by a constant ratio and has  $n$ th term  $ar^{n-1}$ . Check the starting position and keep a surd ratio exact when one is supplied.

$$u_n = a + (n - 1)d, \quad v_n = ar^{n-1}$$

For 5,8,11,... the  $n$ th term is  $3n+2$ , so  $u_8=26$ . For 2,6,18,... the common ratio is 3 and  $u_n=2 \times 3^{n-1}$ , giving  $u_4=54$ . The geometric sequence  $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$  has ratio  $\sqrt{2}$ . These are finite-term calculations; no infinite-series sum is used.

A geometric ratio is not a common difference. The exponent is  $n-1$  when  $a$  is the first term at position 1. A finite pattern does not itself justify an infinite-sum formula.

AQA A23–A25 Higher uses arithmetic rules, geometric progression patterns and quadratic  $n$ th terms. Arithmetic-series formulae and infinite-series sums are excluded from this GCSE lesson.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Algebraic notation, substitution and vocabulary · Foundation

A term is a part joined by addition or subtraction. In  $4+3d$ , 4 is a constant and 3 is the coefficient of  $d$ . The expression has no equality sign;  $4+3d=19$  is an equation. A formula connects named quantities. In  $ab$ , multiplication is understood;  $a^2b$  means  $a \times a \times b$ , not  $a \times b \times b$ .

$$a^2b = (-2)^2 \times 3 = 12$$

For  $d=5$ , the charge is  $4+3 \times 5=19$  yuan. For  $a=-2, b=3$ ,  $a^2b=(-2)^2 \times 3=12$ . The fraction coefficient in  $(3/4)x$  gives 6 when  $x=8$ . An inequality  $4+3d \leq 19$  describes all permitted distances, while an identity such as  $2(x+3)=2x+6$  is true for every  $x$ .

Put a negative substituted number in brackets before squaring. A coefficient is not an exponent. An expression can be evaluated but cannot be solved unless a condition or equation is supplied.

AQA A1–A3 require precise notation and vocabulary, including formulae from other subjects. Identify the inputs and units before substitution; keep fraction coefficients exact.

### Constant + variable charge

4

+

$3 \times d$

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## ■ Collecting, expanding and factorising expressions · Foundation

Collect like terms, distribute over brackets and reverse expansion by factorising. Take out a common factor first. Expand products of two binomials and factorise simple monic quadratics and differences of squares.

$$(x+2)(x+3) = x^2 + 5x + 6$$

$3x+4x=7x$ ;  $3x+4y$  cannot be combined. Expand  $2(x+3)=2x+6$  and  $(x+2)(x+3)=x^2+5x+6$ . Reverse the last identity to factorise  $x^2+5x+6$ . Difference of squares gives  $x^2-9=(x-3)(x+3)$ . The same expansion rule works with given roots:  $(\sqrt{2}+1)(\sqrt{2}-1)=2-1=1$ .

$x^2$  and  $x$  are unlike terms. A factor multiplies a whole expression; it is not a separate added term. Check a proposed factorisation by expanding it.

AQA A4 Foundation includes collecting, single brackets, common factors, two-binomial expansion and monic quadratic factorisation. Higher adds non-monic factorisation, more binomials, surds and algebraic fractions.

## ■ Collecting, expanding and factorising expressions · Higher

Collect only like terms. Distribute multiplication over every term in a bracket, including signs. Factorising reverses expansion. Use common factors first; then pairs for a quadratic. Algebraic fractions may cancel common factors, with forbidden denominator values stated.

$$(x+2)(x+3) = x^2 + 5x + 6$$

$3x+4x=7x$ , but  $3x+4y$  cannot be combined. Expand  $2(x+3)=2x+6$  and  $(x+2)(x+3)=x^2+5x+6$ . Reverse the last result to factorise  $x^2+5x+6$ . Difference of squares gives  $x^2-9=(x-3)(x+3)$ . For Higher,  $2x^2+5x+2=(2x+1)(x+2)$ , and  $(x^2-9)/(x-3)=x+3$  only when  $x \neq 3$ . Also  $(x+1)(x+2)(x+3)=x^3+6x^2+11x+6$ .

$x^2$  and  $x$  are unlike terms. Cancelling across a sum is invalid; factor the whole expression first. A simplified fraction must retain values excluded by its original denominator.

AQA A4 Foundation includes collecting, single brackets, common factors, two-binomial expansion and monic quadratic factorisation. Higher adds non-monic factorisation, more binomials, surds and algebraic fractions.

$$(x+2)(x+3) = x^2 + 5x + 6$$

|   |       |      |
|---|-------|------|
| 3 | $3x$  | 6    |
| x | $x^2$ | $2x$ |
|   | x     | 2    |

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## ■ Rearranging formulae and checking the subject · Foundation

Undo operations on both sides in reverse order. In  $A=bh/2$ , multiply both sides by 2 then divide by nonzero  $h$  to obtain  $b=2A/h$ . When the desired variable occurs twice, collect it before dividing. State restrictions introduced by division.

$$A = \frac{bh}{2} \iff b = \frac{2A}{h} \quad (h \neq 0)$$

For  $A=24, h=6$  in  $A=bh/2$ ,  $b=2 \times 24/6=8$ . From  $v=u+at$ ,  $a=(v-u)/t$ ;  $v=19, u=4, t=5$  gives  $a=3$ . From  $y=3x+2$ ,  $x=(y-2)/3$ . From  $p=qx+r$ ,  $x=(p-r)/q$  for  $q \neq 0$ . Substitute the result into the original equation to check it.

Doing an operation to only one side breaks the equality. The subject is a variable, not a numerical answer. Check the rearranged result by putting the found value into the original formula.

AQA A5 includes standard and given formulae from other subjects, in words and symbols. Choose a valid unit conversion before substituting; a rearrangement does not itself change units.

## ■ Rearranging formulae and checking the subject · Higher

Undo operations on both sides in reverse order. In  $A=bh/2$ , multiply both sides by 2 then divide by nonzero  $h$  to obtain  $b=2A/h$ . When the desired variable occurs twice, collect it before dividing. State restrictions introduced by division.

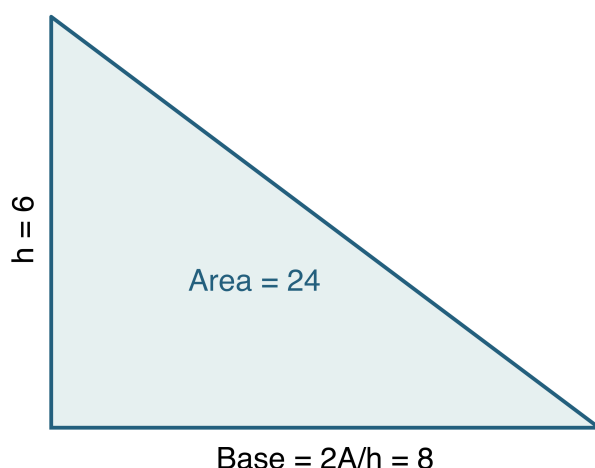
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For  $A=24, h=6$ ,  $b=2 \times 24/6=8$ . From  $v=u+at$ ,  $a=(v-u)/t$ ; for  $v=19, u=4, t=5$  this gives  $a=3$ . From  $y=3x+2$ ,  $x=(y-2)/3$ . From  $p=qx+r$ ,  $x=(p-r)/q$  for  $q \neq 0$ . Higher may factor repeated subjects:  $y=3x+px$  gives  $x=y/(3+p)$  when  $p \neq -3$ .

Doing an operation to only one side breaks the equality. The subject is a variable, not a numerical answer. Check the rearranged result by putting the found value into the original formula.

AQA A5 includes standard and given formulae from other subjects, in words and symbols. Choose a valid unit conversion before substituting; a rearrangement does not itself change units.

## Undo the area formula to find the base



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Identities, equivalence and algebraic arguments · Foundation

An equation may hold only for some values, while an identity holds for every allowed input. Establish equivalent expressions by valid expansion or factorisation. A few matching inputs are checks rather than a general argument.

$$3(x + 2) - x = 2x + 6$$

Expand  $3(x+2)-x=3x+6-x=2x+6$ . This chain of valid steps shows the expressions agree for every  $x$ . However,  $x^2=x$  holds only when  $x=0$  or  $x=1$ ; at  $x=2$ ,  $4 \neq 2$ . Evaluating  $(2n+1)^2$  at  $n=3$  gives 49, but this one calculation does not establish an all-integers claim.

Start from an expression or the assumptions, not from the conclusion as though it were already true. An example can disprove an all-values claim, but one confirming example cannot prove it.

AQA A3/A6 Foundation distinguishes expression, equation and identity and argues equivalence using algebra. General parity and divisibility proofs belong to the Higher variant.

### ■ Identities, equivalence and algebraic arguments · Higher

An equation may be true only at certain values. An identity is true at every allowed value. Establish equivalence by valid expansion or factorisation; testing a few inputs is only a check. Higher proofs use a general integer or algebraic variable and a conclusion tied to its definition.

$$3(x + 2) - x = 2x + 6$$

Expanding  $3(x+2)-x$  gives  $3x+6-x=2x+6$ , proving equivalence for every  $x$ . But  $x^2=x$  holds only for  $x=0$  or  $1$ . A counterexample  $x=2$  rejects an all-values claim. For Higher, an odd integer is  $2n+1$ ; its square is  $4n^2+4n+1=2(2n^2+2n)+1$ , so it is odd for every integer  $n$ .

Start from an expression or the assumptions, not from the conclusion as though it were already true. An example can disprove an all-values claim, but one confirming example cannot prove it.

AQA A6 Foundation distinguishes equation/identity and argues equivalence. Higher extends this to algebraic proofs. State integer restrictions when using parity or consecutive integers.

## Valid expansion establishes equivalence

$$3(x + 2) - x$$

$$3x + 6 - x$$

$$2x + 6$$

*Original native-lesson illustration; labels belong to its worked example.*

### ■ Function machines and reversing operations · Foundation

Follow the operations in their stated order. To recover a starting input, undo the final operation first. A table pairs each input with its output. The same starting input must have only one output for the rule to be a function.

$$x \mapsto 2x + 3$$

Input 4 gives  $2 \times 4 + 3 = 11$ . To find the unknown input for output 11, subtract 3 to get 8 then divide by 2 to get 4. Inputs -1,0,1 give outputs 1,3,5. A table lists each input beside its output; the operations are always performed in the same order.

Reversing the rule does not mean repeating it. The last forward step is the first reverse step. Different operation orders can produce different outputs.

AQA A7 Foundation interprets simple functions as input-output rules. Higher also uses formal inverse and composite function notation in the separate function lesson.

### ■ Function machines and reversing operations · Higher

Follow the operations in their stated order. To recover a starting input, undo the final operation first. A table pairs each input with its output. The same starting input must have only one output for the rule to be a function.

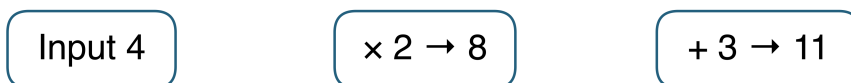
$$x \mapsto 2x + 3$$

Input 4 gives  $2 \times 4 + 3 = 11$ . To reverse output 11, subtract 3 to get 8 then divide by 2 to get 4. Inputs -1,0,1 give outputs 1,3,5. If a second machine squares its input, passing 4 through the first then the second gives  $11^2 = 121$ ; reversing their order gives  $2 \times 16 + 3 = 35$ .

Reversing the rule does not mean repeating it. The last forward step is the first reverse step. Different operation orders can produce different outputs.

AQA A7 Foundation interprets simple functions as input-output rules. Higher also uses formal inverse and composite function notation in the separate function lesson.

**Forward order; reverse by undoing the last step first**



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## ■ Domains, inverses and composition · Higher

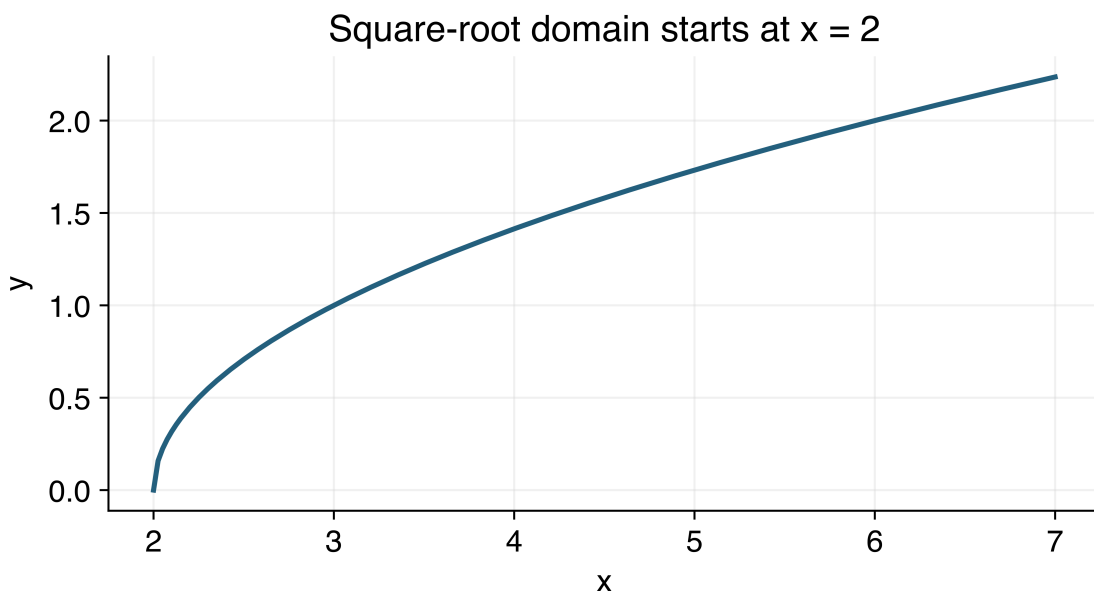
State the domain and range. For an inverse, first ensure the function is one-to-one on its domain. Composition  $fg$  means apply  $g$  first, then  $f$ ; the intermediate output must be an allowed input to  $f$ .

$$f(g(x)) = (f \circ g)(x), \quad f^{-1}(f(x)) = x$$

For  $f(x)=\sqrt{x-2}$ ,  $x \geq 2$  and the range is  $y \geq 0$ . From  $y=\sqrt{x-2}$ ,  $x=y^2+2$ . Thus  $f$  inverse( $x$ )= $x^2+2$  with  $x \geq 0$ . For  $g(x)=x+3$ ,  $fg(1)=f(4)=\sqrt{2}$ .

Squaring can introduce extraneous solutions. Restricting a parabola's domain is essential before claiming an inverse. A horizontal translation inside  $f$  has the opposite sign to the graph's movement.

Check  $f(f \text{ inverse}(x))=x$  on the inverse domain. Use a sketch to test whether a horizontal line meets the original graph more than once.



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## ■ Coordinate quadrants and basic graph families · Foundation

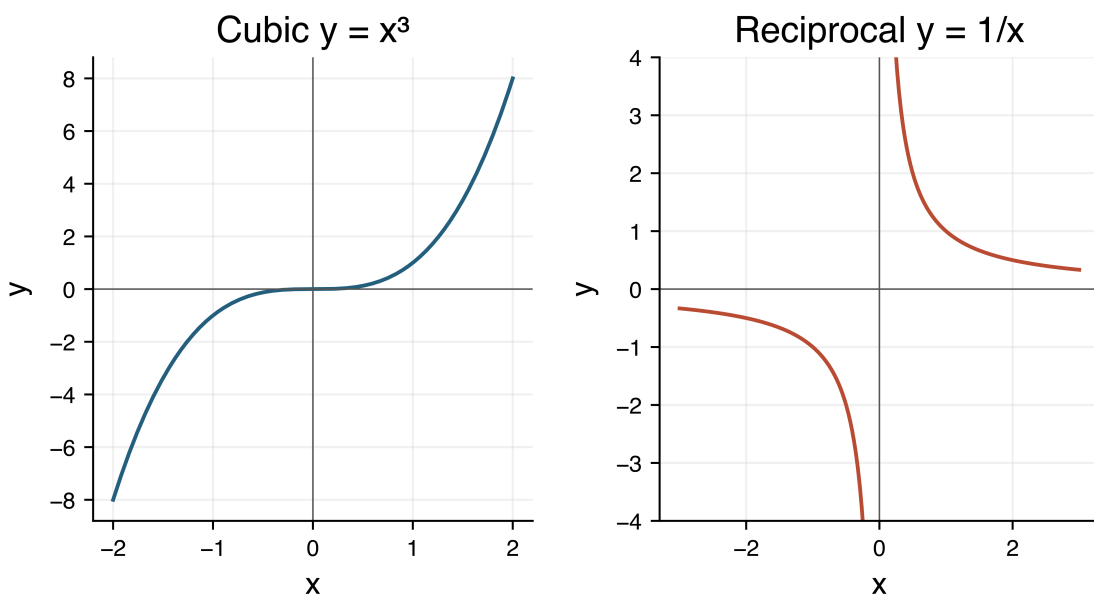
Coordinates are ordered (x,y). Signs identify the four quadrants. Use a value table, intercepts and symmetry to sketch a line, quadratic, cubic or reciprocal. For  $y=1/x$ , zero is excluded and the axes are asymptotes. Approximate intersections give graphical solutions.

$$y = x^3, \quad y = \frac{1}{x} \quad (x \neq 0)$$

For  $y=x^3$ , inputs -2,-1,0,1,2 give -8,-1,0,1,8. For  $y=1/x$ , inputs -2,-1,1,2 give -1/2,-1,1,1/2; never substitute zero. The point (-2,3) is in quadrant II. The graphs  $y=x^2$  and  $y=4$  meet at  $x=-2$  and  $x=2$ . A quadratic  $y=(x-3)(x-7)$  has roots 3,7 and symmetry line  $x=5$ .

Join a reciprocal branch smoothly on its own side of zero; never draw a segment through the excluded input. A graph table needs enough points to reveal a turning point or shape.

AQA A8/A11/A12 Foundation includes all quadrants, lines, quadratics, simple cubics and reciprocals. Higher exponential and degree-based trigonometric families are treated separately.



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## ■ Exponential and degree-based trigonometric graphs · Higher

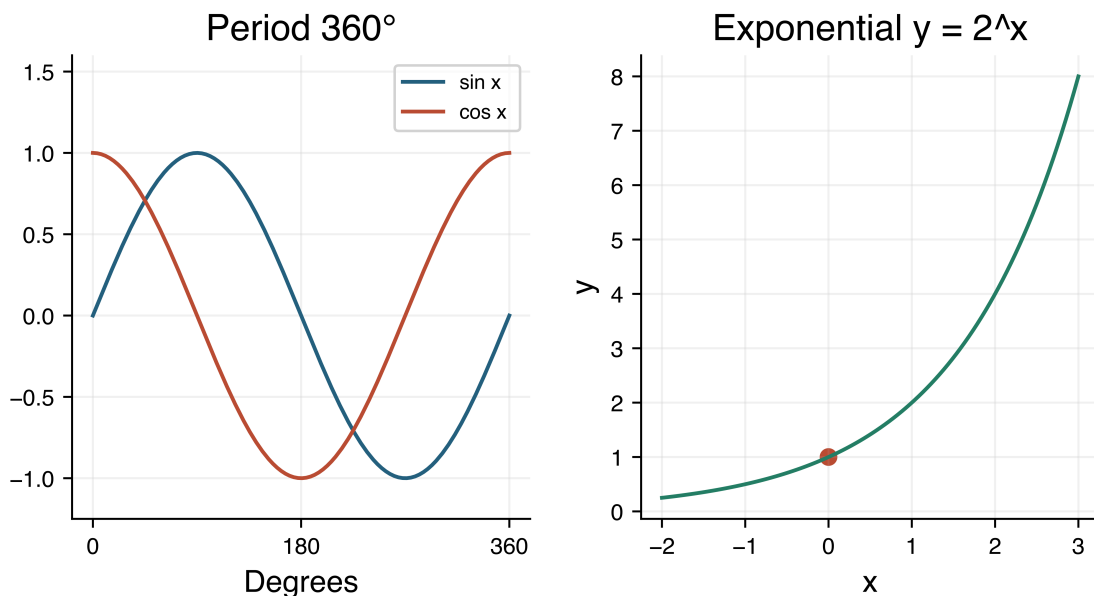
For  $y=k^x$  with  $k>0, k\neq 1$ , the y-intercept is 1;  $k>1$  gives growth and  $0<k<1$  gives decay. In degrees, sine and cosine repeat every  $360^\circ$  with range  $[-1,1]$ ; tangent repeats every  $180^\circ$  and is undefined at  $90^\circ+180^\circ n$ . Label axes in degrees.

$$\sin 270^\circ = -1, \quad \tan 45^\circ = 1$$

For  $y=2^x$ ,  $x=-1,0,1,3$  give  $1/2,1,2,8$ . Sine has values 0,1,0,-1,0 at  $0^\circ,90^\circ,180^\circ,270^\circ,360^\circ$ . Cosine starts at 1;  $\cos 180^\circ=-1$ . Tangent has  $\tan 45^\circ=1$  but no finite value at  $90^\circ$ . For  $k=1/2$ , increasing  $x$  produces decay, never a negative output.

Degrees and radians are different units. A steep tangent branch is not a finite point at its asymptote. Exponential growth is not repeated addition of a constant amount.

AQA A12 Higher requires exponential functions with positive bases and sine/cosine/tangent for angles of any size in degrees. No differentiation or radian sector formula is introduced here.



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### ■ Translations and reflections of function graphs · Higher

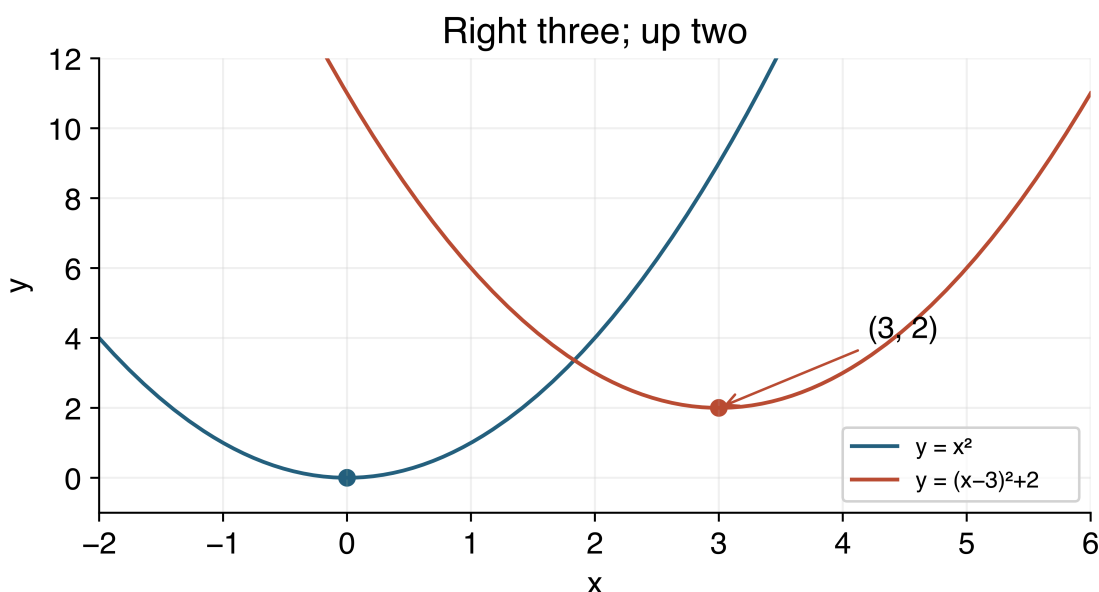
$y=f(x)+a$  shifts the output up by  $a$ .  $y=f(x-a)$  shifts the graph right by  $a$ .  $y=-f(x)$  reflects in the  $x$ -axis;  $y=f(-x)$  reflects in the  $y$ -axis. Transform the points as well as the formula, checking a known feature.

$$y = (x - 3)^2 + 2$$

With  $f(x)=x^2$ ,  $y=f(x-3)+2=(x-3)^2+2$  has vertex  $(3,2)$ . The point  $(1,1)$  on  $f$  moves to  $(4,3)$ . At  $x=4$  the new output is 3. The reflection  $y=-x^2$  turns the minimum at the origin into a maximum. For a nonsymmetric graph,  $f(-x)$  mirrors each  $x$ -coordinate, not each  $y$ -coordinate.

A positive number added inside the input,  $f(x+3)$ , moves the graph left, not right. A reflection in the  $y$ -axis changes  $x$ ; a reflection in the  $x$ -axis changes  $y$ .

AQA A13 Higher covers translations and reflections of a given function. Use features and transformed points to justify the sketch; do not substitute a geometric enlargement for this graph transformation.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Distance-time graphs and contextual intersections · Foundation

Read the axis quantities and units before interpreting shape. On a distance-time graph, slope is speed for a segment with increasing distance; a horizontal segment means no distance change. An intersection of two charge graphs gives equal costs. A curved section requires a local rather than one fixed slope.

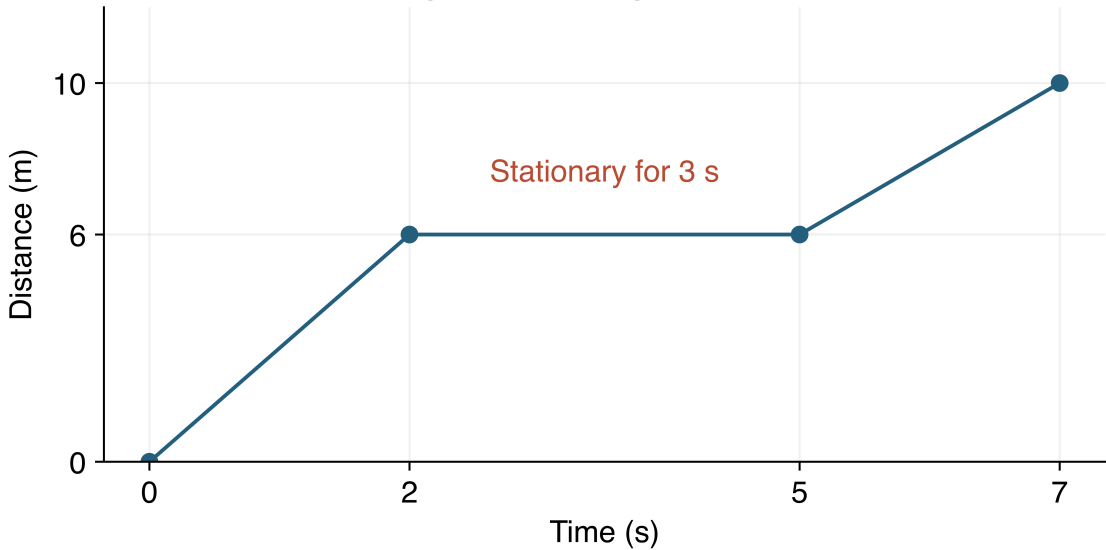
$$v = \frac{\Delta d}{\Delta t}$$

The distance points (0,0), (2,6), (5,6), (7,10), in seconds and metres, give speed  $6/2=3$  m/s for the first section, a 3-second stop, then speed  $(10-6)/(7-5)=2$  m/s. Average speed for the full interval is  $10/7$  m/s, including the stop. Charges  $20+3x$  and  $44+x$  meet at  $x=12$ , at cost 56. If speed increases from 0 to 10 m/s over 5 seconds, its average acceleration is  $10/5=2$  m/s<sup>2</sup>; a speed-time graph measures this through its slope.

A horizontal distance graph does not mean fast motion. A graph height gives distance, while slope gives its rate. Average speed includes every elapsed interval, including waiting.

AQA A14 uses real contexts and graphical solutions. Foundation can interpret a plotted non-standard function; Higher also interprets exponential models and nonlinear graph estimates in the next lesson.

## Slope changes; a flat segment records a stop



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Graphical gradients and area estimates · Higher

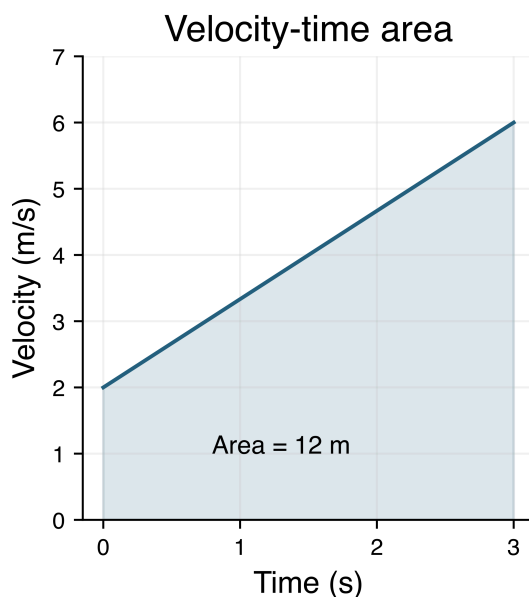
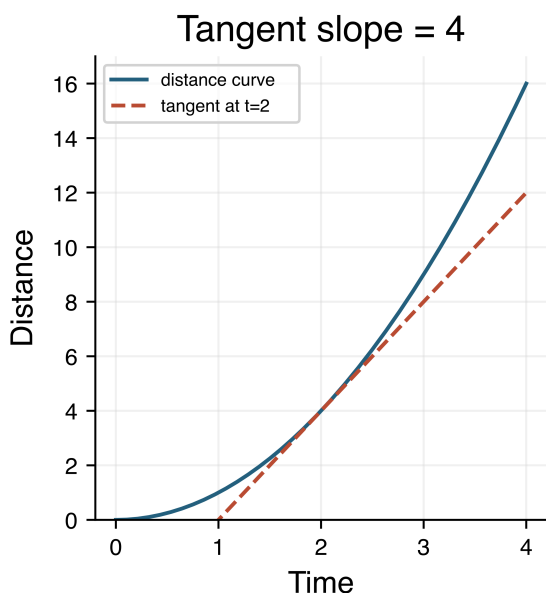
A chord gives an average rate between two inputs; a tangent estimates the local rate. Read two well-separated points on the drawn tangent to reduce measurement error. On a velocity-time graph, area represents displacement. Split it into triangles/trapezia or estimate curved area using narrow strips.

$$\frac{9-1}{3-1} = 4, \quad A = \frac{2+6}{2} \times 3 = 12$$

For a distance curve  $d=t^2$ , the chord from (1,1) to (3,9) has gradient  $(9-1)/(3-1)=4$ . A tangent at (2,4) passes through (1,0) and (3,8), giving gradient 4 without calculus. A velocity-time trapezium with endpoint velocities 2 and 6 m/s over 3 s has area  $(2+6) \times 3/2 = 12$  m. Smaller strips can improve a curved-area estimate; curvature affects over/underestimation. A cost-versus-quantity tangent through (2,12) and (6,28) gives a local rate  $(28-12)/(6-2)=4$  yuan per extra item, linking graph slope to a financial interpretation.

Tangent estimates and curve chords use different pairs of points. Area under a distance-time graph is not distance travelled. A strip estimate is approximate unless the graph is linear on each strip.

AQA A15 and R15 Higher require graph-based rates and area interpretation. This lesson uses graphical reasoning and geometric areas, not symbolic differentiation/integration.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Origin-centred circles and tangent equations · Higher

For an origin-centred circle,  $x^2 + y^2 = r^2$ . A point is on the circle if its squared coordinates sum to  $r^2$ . A tangent is perpendicular to the radius there. Use a negative reciprocal gradient when both gradients are finite; handle horizontal/vertical cases directly.

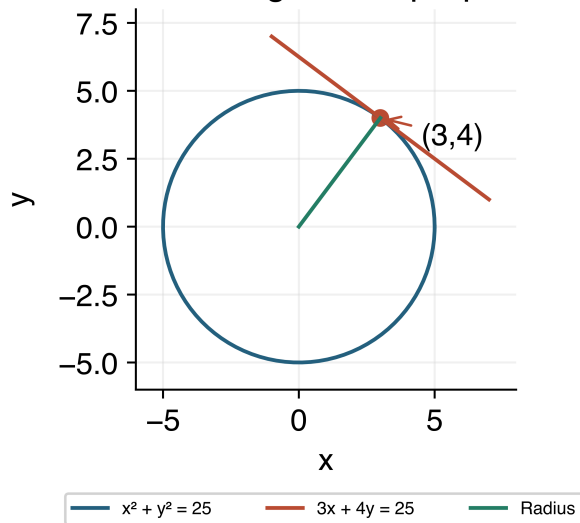
$$x^2 + y^2 = 25, \quad 3x + 4y = 25$$

At (3,4),  $r^2 = 3^2 + 4^2 = 25$ , so  $x^2 + y^2 = 25$ . The radius gradient is  $4/3$ , hence tangent gradient  $-3/4$ . Its equation  $y - 4 = (-3/4)(x - 3)$  simplifies to  $3x + 4y = 25$ . At (5,0) the radius is horizontal and the tangent is the vertical line  $x = 5$ .

The radius and tangent share a point but different directions. Do not use the negative reciprocal of zero; a horizontal radius has a vertical tangent. Check the tangent passes through its contact point.

AQA A16 Higher concerns circles centred at the origin. Offset-centre circle formulae and circle calculus are not needed for this lesson.

## Radius and tangent are perpendicular



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Numerical iteration for equation roots · Higher

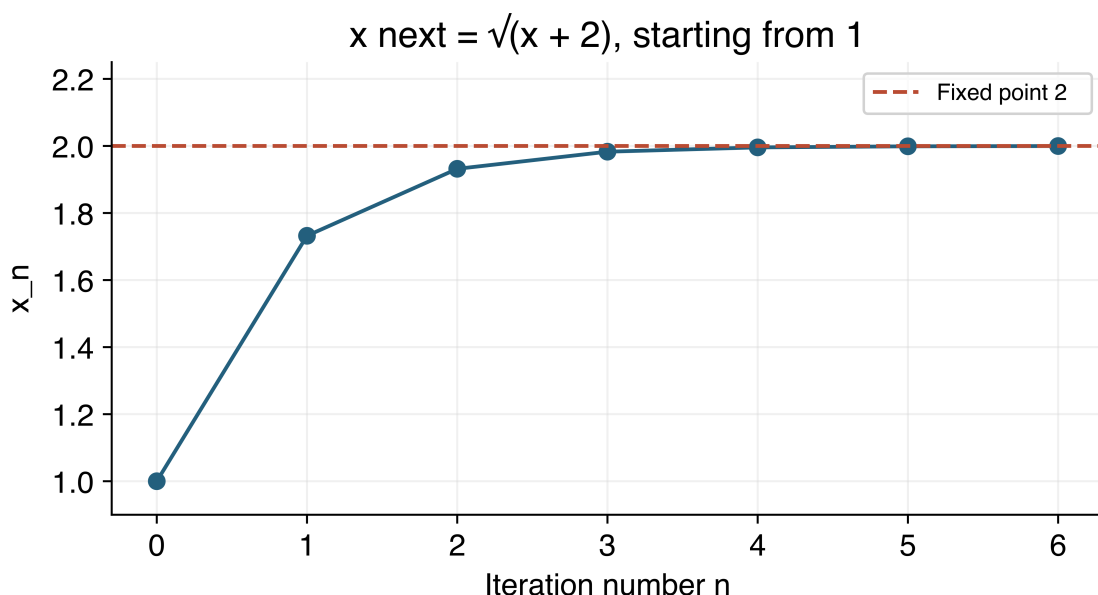
Rearrange an equation as  $x=g(x)$ , choose  $x_0$  and use  $x_{n+1}=g(x_n)$ . Record sufficient working precision. A stable-looking sequence must still be checked in the original equation; not every rearrangement converges. A sign change across continuous inputs can check a rounded root.

$$x_{n+1} = \sqrt{x_n + 2}, \quad x_0 = 1$$

For  $x^2-x-2=0$ , use  $x_{\text{next}}=\sqrt{x+2}$  with  $x_0=1$ :  $x_1=\sqrt{3}\approx 1.73205$ ,  $x_2\approx 1.93185$ ,  $x_3\approx 1.98289$ . The positive fixed point is 2 because  $2=\sqrt{4}$  and  $2^2-2-2=0$ . The rearrangement only seeks a nonnegative root; the original equation also has root -1. Using  $x_{\text{next}}=x^2-2$  from 3 instead gives 7 then 47, so a different rearrangement can diverge.

Use the previous iterate, not the starting value every time. Keep more digits than the final answer. A sign change needs continuity; a jump across a vertical asymptote does not guarantee a root.

AQA A20 Higher uses numerical iteration and suffix notation. Use the calculator to follow the stated rule, then report the requested rounding with a check. Newton derivatives are outside this GCSE lesson.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Linear inequalities and number-line solutions · Foundation

Solve a linear inequality using the same balance operations as an equation. Multiplying or dividing by a negative reverses its direction. Use a closed endpoint for  $\leq$  or  $\geq$  and an open endpoint for  $<$  or  $>$ . Intersect restrictions to find their common permitted inputs.

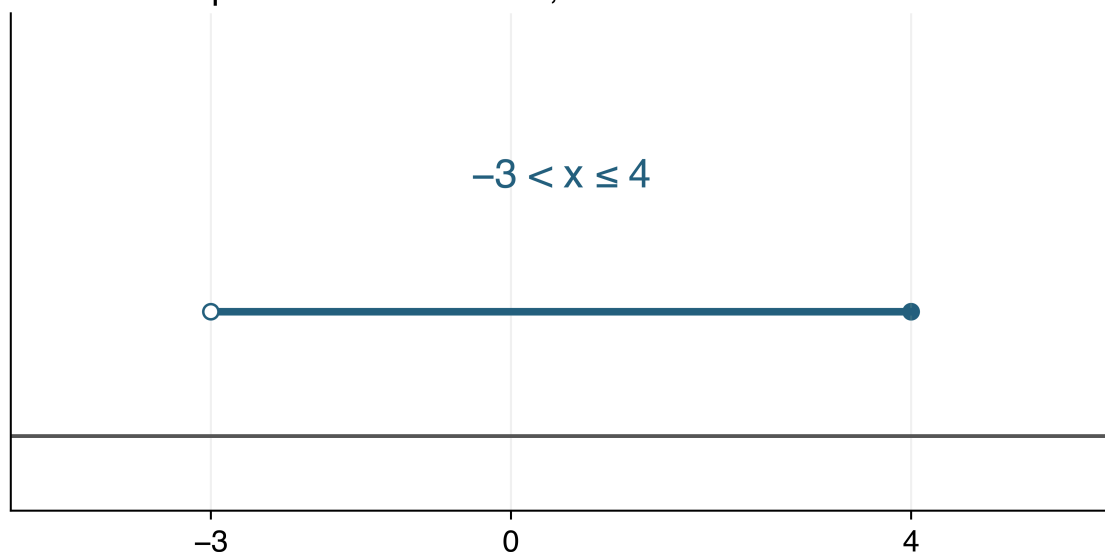
$$3x + 5 \leq 17 \iff x \leq 4$$

$3x+5 \leq 17$  gives  $x \leq 4$ . For nonnegative whole items, the permitted values are 0,1,2,3,4. Also  $-2x < 6$  gives  $x > -3$  after division by  $-2$ . The combined restriction  $-3 < x \leq 4$  has an open circle at  $-3$  and a closed circle at  $4$ . A value  $x=5$  fails the original budget because  $3 \times 5 + 5 = 20$ .

A reversed sign is needed only when multiplying or dividing by a negative, not when adding a negative. Include the physical domain: negative or fractional item counts may be meaningless.

AQA A22 Foundation requires one-variable linear inequalities and number lines. Higher quadratic and two-variable regions are in a separate lesson; avoid replacing the inequality with a single boundary value.

Open means excluded; closed means included



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Quadratic inequalities and two-variable regions · Higher

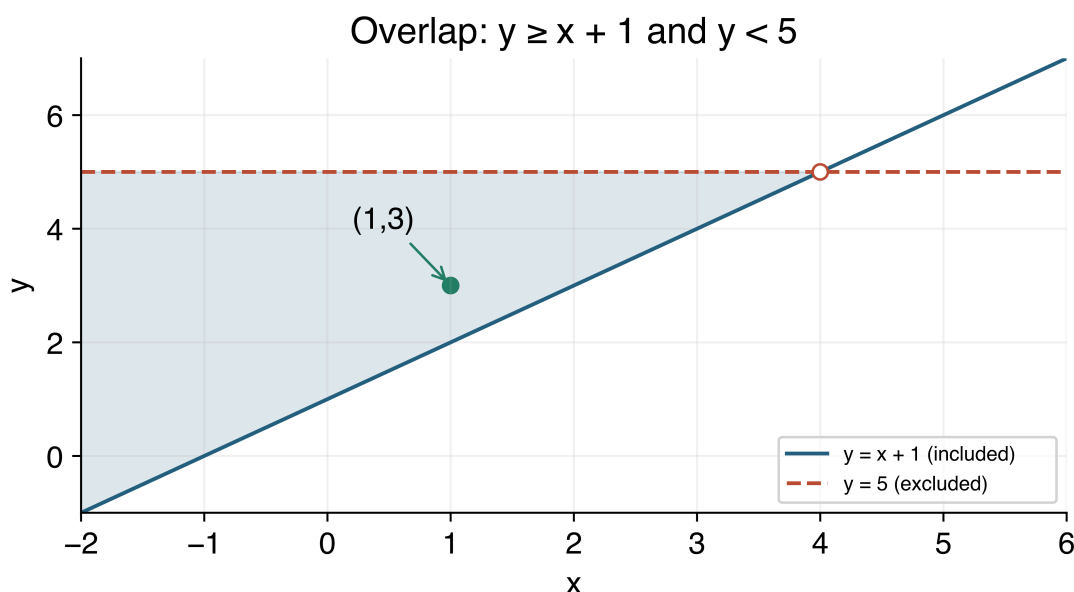
For a quadratic inequality, locate roots and test the sign in each interval. For a two-variable inequality, draw its equality boundary; use a dashed line if equality is excluded and a solid line if included. Test a point on each side and intersect the allowed regions. Write the domain or set notation clearly.

$$(x - 3)(x - 7) \leq 0 \iff x \in [3, 7]$$

$(x-3)(x-7) \leq 0$  holds on  $3 \leq x \leq 7$ , since the factors have opposite signs between the roots and the endpoints give zero. For  $y \geq x+1$  and  $y < 5$ , shade on/above the solid line  $y=x+1$  and below the dashed line  $y=5$ . Their meeting input is  $x=4$ , but  $(4,5)$  is excluded by  $y < 5$ . The point  $(1,3)$  satisfies both inequalities.

For an upward-opening quadratic, positive values lie outside the roots, not between them. A boundary intersection is not necessarily included. Use a test point not on the boundary itself.

AQA A22 Higher includes quadratic inequalities in one variable and linear inequalities in two variables. The graphical overlap is the solution region, not one selected point.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Figurate, geometric and Fibonacci-type sequences · Foundation

Record positions and terms separately. Triangular numbers add 1,2,3,...; square and cube numbers use  $n^2$  and  $n^3$ . A geometric sequence multiplies by a common positive ratio. A Fibonacci-type sequence starts with stated terms and then adds its two predecessors. A supplied recursive rule must include enough initial values.

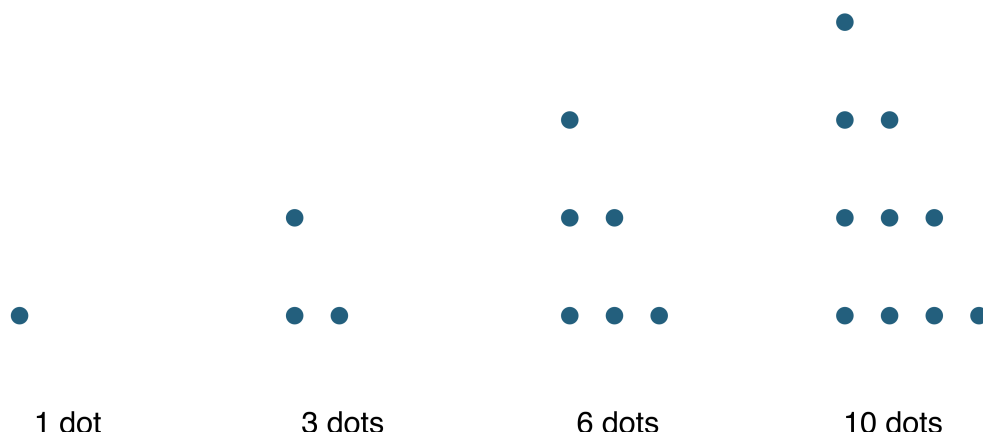
$$u_n = 2 \times 3^{n-1}$$

Triangular terms are 1,3,6,10,15; square terms 1,4,9,16,25; cube terms 1,8,27,64,125. The geometric sequence 2,6,18,54,... has ratio 3 and  $n$ th term  $2 \times 3^{(n-1)}$ . Starting 2,3 and adding the previous two gives 2,3,5,8,13. For  $u_{\text{next}} = 2u + 1$  from  $u_1 = 1$ , the next terms are 3,7,15.

A nonconstant first difference does not mean a pattern is random. A geometric ratio is not a common difference. Check the starting index when writing a position rule.

AQA A23/A24 Foundation includes these patterns and positive rational geometric ratios. Higher also permits surd ratios and derives quadratic  $n$ th terms in the next lesson. Infinite-series sums are not part of this GCSE lesson.

## Triangular patterns add a new row each time



*Original native-lesson illustration; labels belong to its worked example.*

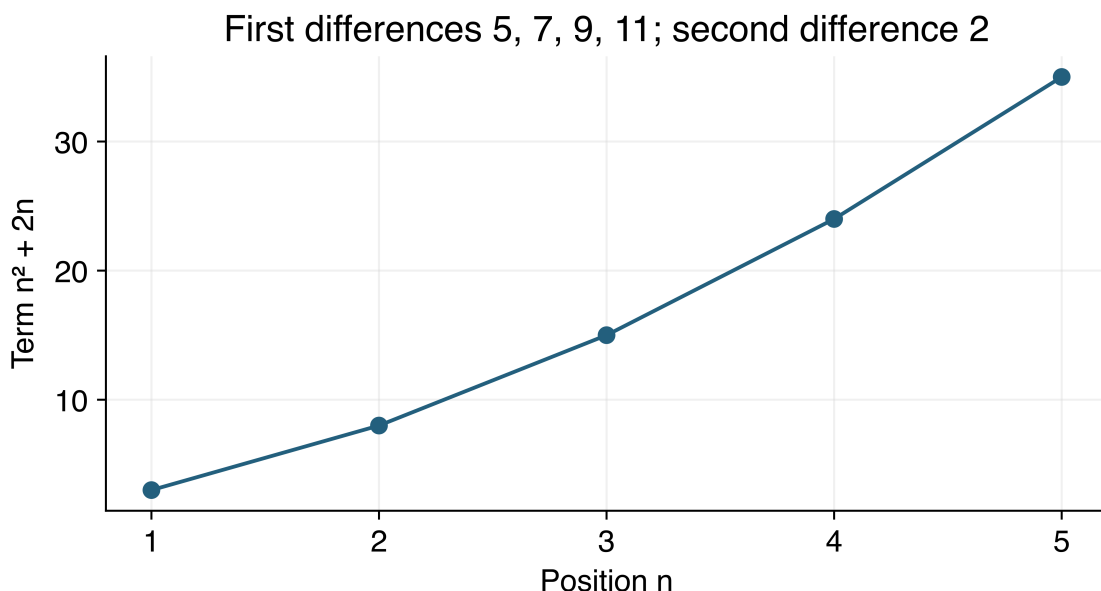
### ■ Quadratic nth terms and surd-ratio progressions · Higher

A constant second difference signals a quadratic rule  $an^2 + bn + c$ . Its value is  $2a$ . Subtract  $an^2$  from the terms; fit the remaining linear rule and test at several positions. A geometric sequence with a surd ratio still multiplies by the same exact factor; keep root values exact.

$$u_n = n^2 + 2n, \quad \Delta^2 u_n = 2$$

For 3,8,15,24, first differences are 5,7,9 and second difference 2, so  $a = 1$ . Subtract  $n^2$  at  $n=1,2,3,4$  to get 2,4,6,8= $2n$ . Thus  $u_n = n^2 + 2n$  and  $u_5 = 35$ . For  $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$  the ratio is  $\sqrt{2}$  and  $u_n = (\sqrt{2})^{(n-1)}$ . Check  $u_3 = 2$  rather than rounding the root repeatedly. The second difference equals  $2a$ , not  $a$ . A quadratic rule must be checked against the first terms; there can be a nonzero constant  $c$ . A finite pattern alone does not prove a unique rule without the stated sequence family.

AQA A25 Higher derives quadratic nth terms; A24 permits surd-ratio sequences. Foundation recognises and generates quadratic patterns without this general derivation.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Algebraic fractions and excluded values · Higher

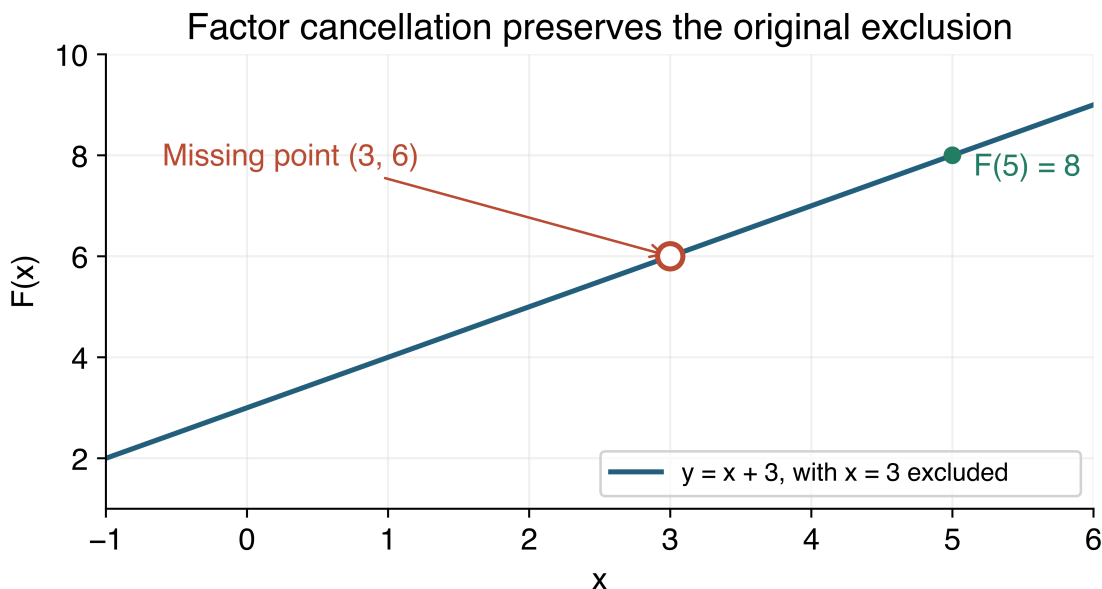
Factor every numerator and denominator before cancelling a common factor. A term joined by addition is not a cancellable factor. Record exclusions from the original expression first. For addition or subtraction use a common denominator, retaining brackets around the entire numerator. Multiply factored numerators and denominators. To divide, multiply by the reciprocal and exclude inputs making the divisor zero as well as inputs making any original fraction undefined.

$$\frac{x^2 - 9}{x - 3} = x + 3 \quad (x \neq 3)$$

For  $F(x) = (x^2 - 9)/(x - 3)$ , factor  $x^2 - 9 = (x - 3)(x + 3)$ . Thus  $F(x) = x + 3$  for  $x \neq 3$ ;  $F(5) = 8$ , but  $F(3)$  is undefined, not 6. For addition,  $1/(x - 1) + 2/(x + 1) = [(x + 1) + 2(x - 1)]/[(x - 1)(x + 1)] = (3x - 1)/(x^2 - 1)$ , with  $x \neq \pm 1$ . At  $x = 3$ , both forms give 1. For subtraction, the numerator is  $(x + 1) - 2(x - 1) = 3 - x$ , so the minus sign acts on both terms. Multiplication  $[(x - 1)/(x + 2)] \times [(x + 2)/(x + 1)]$  gives  $(x - 1)/(x + 1)$ , but the original excludes  $x = -2$  and  $x = -1$ . Division  $[(x^2 - 1)/(x^2 + 3x + 2)] \div [(x - 1)/(x + 2)]$  gives 1, yet  $x = -2, -1, 1$  are excluded:  $x = 1$  makes the divisor zero. Finally  $F(x) = 6$  reduces to  $x + 3 = 6$ . Its only candidate  $x = 3$  is forbidden, so this equation has no solution. Check candidates in the original equation before reporting them.

Never cancel the  $x$  in  $(x + 2)/x$ . A simplified denominator does not show all original restrictions. In a division task, check when the divisor is zero. Cross-multiplication can produce an excluded candidate; a formal root is not automatically a solution.

AQA A4 Higher includes algebraic fractions and valid cancellation. The equation example checks whether a candidate is inside the original domain. Explain each factor cancellation, retain original restrictions and check every equation candidate. These operations extend the existing manipulation example into a complete worked arithmetic sequence.



*Original native-lesson illustration; labels belong to its worked example.*

## Original independent transfer

### ■ Foundation

A taxi charges a fixed 8 CNY plus 3 CNY per kilometre. Form a cost equation. A trip costs 35 CNY; find its distance and check. Then solve the simultaneous equations  $2x + y = 11$ ,  $x - y = 1$ .

### ■ Foundation worked solution

For distance  $d$  in kilometres and cost  $C$  in CNY,  $C = 8 + 3d$ . Rearranging gives  $d = (C - 8)/3 = (35 - 8)/3 = 9$  km. Substitution gives  $8 + 3(9) = 35$  CNY. Adding the simultaneous equations gives  $3x = 12$ , so  $x=4$  and  $y=3$ . Both equations check:  $2(4) + 3 = 11$ ,  $4 - 3 = 1$ . The cost intercept represents the charge at zero distance, not the distance for zero cost.

## ■ Higher

Solve  $(x^2 - 9)/(x - 3) = x^2 - 5$  on its original real domain. Then solve  $x^2 - 4x - 5 \leq 0$  completely. State the turning point and roots of  $y = x^2 - 4x - 5$ .

## ■ Higher worked solution

The equation excludes  $x=3$ . Cancellation gives  $x + 3 = x^2 - 5$ , hence  $x^2 - x - 8 = 0$  and  $x = (1 \pm \sqrt{33})/2$ . Neither is 3, so both remain. For the separate inequality,  $(x - 5)(x + 1) \leq 0$  gives  $-1 \leq x \leq 5$ , including both zero endpoints. Completing the square gives  $y = (x - 2)^2 - 9$ ; its turning point is (2,-9), a minimum, with roots -1 and 5. Do not confuse the two different quadratics.

## Terms

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|                       |   |
|-----------------------|---|
| identity              | . |
| root                  | . |
| discriminant          | . |
| elimination           | . |
| gradient              | . |
| common difference     | . |
| common ratio          | . |
| coefficient           | . |
| factorise             | . |
| subject               | . |
| equivalent expression | . |
| input                 | . |
| domain                | . |
| asymptote             | . |
| period                | . |
| graph translation     | . |
| stationary            | . |
| tangent gradient      | . |
| radius gradient       | . |
| iteration             | . |
| solution set          | . |
| boundary line         | . |
| recursive rule        | . |
| second difference     | . |
| excluded value        | . |