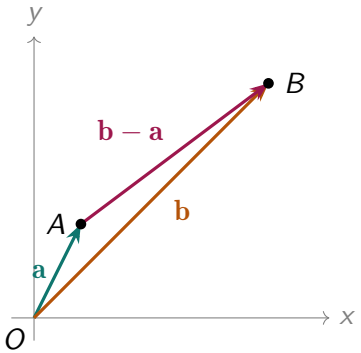
Vectors

AQA A-level Mathematics 7357, section J: vectors in 2D and 3D, magnitude and direction, position vectors, geometry, forces

Writing a vector

Idea	Notation
Column vector	$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\mathbf{i} + 3\mathbf{j}$
In 3D	$\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$
Magnitude	$\lvert x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\rvert = \sqrt{x^2 + y^2 + z^2}$
Displacement $A \to B$	$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

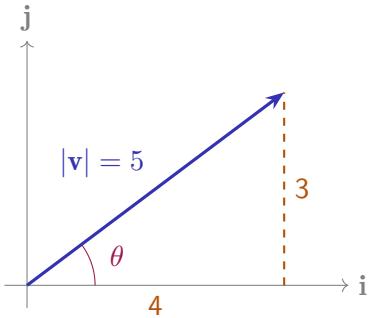
Displacement is end minus start



- $A(1,2)$, $B(5,5)$.
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (4,3)$.
- $|\overrightarrow{AB}| = \sqrt{16+9} = 5$; $\overrightarrow{BA} = (-4,-3)$.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

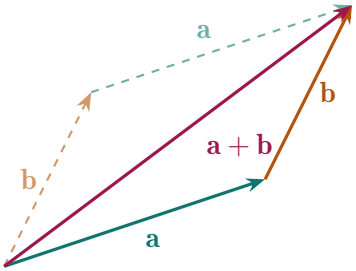
Magnitude and direction



- $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$: $|\mathbf{v}| = 5$.
- Angle with $\mathbf{i}$: $\tan\theta = \frac{3}{4}$, $\theta = 36.9°$.
- 10 at $30°$:
  $(10\cos 30°, 10\sin 30°) = (8.66, 5)$.

$$|\mathbf{v}| = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}$$

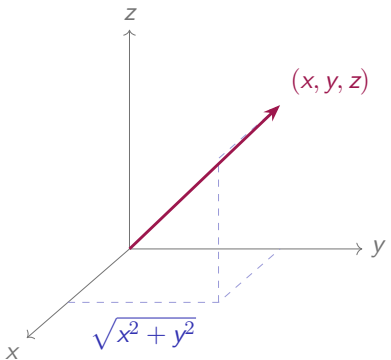
Adding vectors nose to tail



- $\mathbf{a} = (3, 1)$, $\mathbf{b} = (1, 2)$.
- $\mathbf{a} + \mathbf{b} = (4, 3)$: add components.
- $2\mathbf{a} = (6, 2)$ is parallel to $\mathbf{a}$.

Order does not matter: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, the two routes round the parallelogram.

Vectors in three dimensions



- $A(1, 2, 3)$, $B(4, 6, 15)$.
- $\overrightarrow{AB} = (3, 4, 12)$.
- $|\overrightarrow{AB}| = \sqrt{9 + 16 + 144} = 13$.

$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$$

Unit vectors

1. **Vector** $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$, $|\mathbf{v}| = 13$
2. **Divide** $\hat{\mathbf{v}} = \frac{1}{13}(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$
3. **Check** $\frac{9+16+144}{169} = 1$ ✓
4. **Use** speed 26 in that direction: $26\hat{\mathbf{v}} = 6\mathbf{i} + 8\mathbf{j} + 24\mathbf{k}$

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

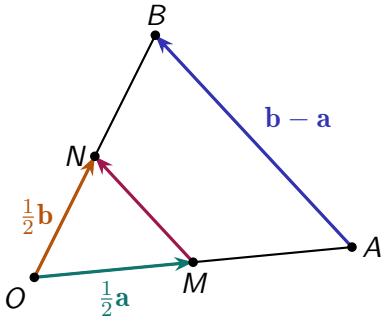
Proving three points collinear

1. **Points** $A(1,1)$, $B(3,4)$, $C(7,10)$
2. **Vectors** $\overrightarrow{AB} = (2,3)$, $\overrightarrow{BC} = (4,6)$
3. **Parallel** $\overrightarrow{BC} = 2\overrightarrow{AB}$
4. **Shared** both pass through B, so A, B, C are **collinear**

Decision

Parallel vectors at different places only show parallel lines.

A proof by vectors: the midpoint theorem



- M, N midpoints of OA, OB.
- $\overrightarrow{MN} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{b}-\mathbf{a})$.
- MN is parallel to AB and half its length.

A point dividing a line in a ratio

1. **Route** $\mathbf{p} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$
2. **Simplify** $\mathbf{p} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$
3. **Numbers** $A(1,2)$, $B(7,5)$: $\mathbf{p} = (1,2) + \frac{1}{3}(6,3) = (3,3)$

Decision

Check: $AP = \sqrt{5}$, $PB = 2\sqrt{5}$, a ratio of $1 : 2$ ✓

Forces are vectors

1. **Given** $\mathbf{F}_1 = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{F}_2 = -\mathbf{i} + 5\mathbf{j}$ N
2. **Resultant** $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = 2\mathbf{i} + 3\mathbf{j}$ N
3. **Size** $|\mathbf{R}| = \sqrt{13} = 3.61$ N
4. **Direction** $\tan\theta = \frac{3}{2}$: 56.3° above $\mathbf{i}$
5. **Acceleration** $m = 2$ kg: $\mathbf{a} = \frac{\mathbf{R}}{m} = \mathbf{i} + 1.5\mathbf{j}$ m s^{-2}

Preview: the scalar product

1. **Product** $\mathbf{a} = (4, 1)$, $\mathbf{b} = (1, 3)$: $\mathbf{a} \cdot \mathbf{b} = 4 + 3 = 7$
2. **Sizes** $|\mathbf{a}| = \sqrt{17}$, $|\mathbf{b}| = \sqrt{10}$
3. **Angle** $\cos\theta = \frac{7}{\sqrt{170}} = 0.537$, $\theta = 57.5°$

Decision

$\mathbf{a} \cdot \mathbf{b} = 0$ would mean perpendicular; here it is 7.

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Traps with vectors

Slips

$\overrightarrow{AB} = \mathbf{a} - \mathbf{b}$.

$|3\mathbf{i} + 4\mathbf{j}| = 7$.

Parallel vectors taken as proof of collinearity.

Fixes

End minus start: $\mathbf{b} - \mathbf{a}$.

Pythagoras: $\sqrt{9 + 16} = 5$.

Parallel and a common point.

Check yourself

Question	Answer
$\overrightarrow{PQ}$ for $P(2,-1)$, $Q(5,3)$?	$(3,4)$
$\lvert 2\mathbf{i}-\mathbf{j}+2\mathbf{k}\rvert$?	3
Unit vector along $6\mathbf{i}+8\mathbf{j}$?	$0.6\mathbf{i}+0.8\mathbf{j}$
Is $(2,-6)$ parallel to $(-1,3)$?	yes: -2 times
Midpoint of $\mathbf{a}$ and $\mathbf{b}$?	$\frac{1}{2}(\mathbf{a}+\mathbf{b})$