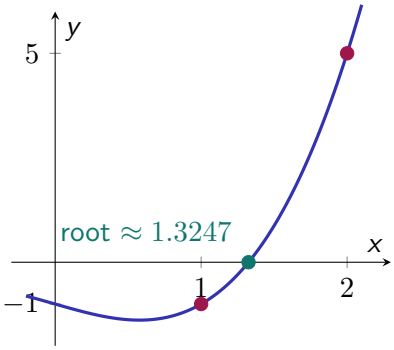
Numerical methods

AQA A-level Mathematics 7357, section I: sign change, bisection, iteration, Newton-Raphson, trapezium rule

When algebra cannot finish the job

Method	**Does**	**Rule**
Sign change	locates a root	**continuous** f, $f(a)f(b) < 0$
Iteration	refines a root	$x_{n+1} = g(x_n)$
Newton–Raphson	refines a root fast	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
Trapezium rule	estimates an integral	$\frac{h}{2}\left[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right]$

A sign change locates a root



- $f(x) = x^3 - x - 1$.
- $f(1) = -1 < 0$, $f(2) = 5 > 0$.
- f is **continuous**, so a root lies in $(1, 2)$.

When a sign change misleads

Change, no root

$f(x) = \frac{1}{x}$: $f(-1) = -1$, $f(1) = 1$.

The graph jumps at $x = 0$: an asymptote, not a root.

Root, no change

$f(x) = (x-1)^2$: $f(0) = 1$, $f(2) = 1$.

A repeated root at $x = 1$ touches the axis.

Two roots close together can also hide inside one interval.

Bisection, step by step

Interval	**Midpoint** m	$f(m)$	**Keep**
$[1, 2]$	1.5	0.875	$[1, 1.5]$
$[1, 1.5]$	1.25	-0.297	$[1.25, 1.5]$
$[1.25, 1.5]$	1.375	0.225	$[1.25, 1.375]$
$[1.25, 1.375]$	1.3125	-0.0515	$[1.3125, 1.375]$

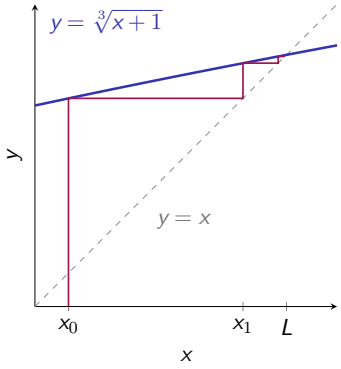
Proving a root to 4 decimal places

1. **Claim** root $= 1.3247$ to 4 d.p.
2. **Bounds** test 1.32465 and 1.32475
3. **Values** $f(1.32465) = -0.00029 < 0$, $f(1.32475) = 0.00014 > 0$
4. **Conclude** sign change and f **continuous**: root in $(1.32465, 1.32475)$

Decision

Every number in that interval rounds to 1.3247.

Iteration as a staircase



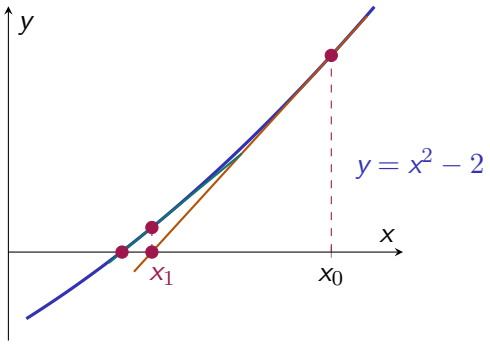
- $x^3 - x - 1 = 0 \iff x = \sqrt[3]{x+1}$.
- $x_0 = 1$: 1.2599, 1.3123, 1.3224, 1.3243.
- **Converges** to 1.3247.

A rearrangement that diverges

Rearrangement	$\lvert g'(1.3247)\rvert$	**From** $x_0 = 1.5$
$g(x) = \sqrt[3]{x+1}$	$0.19 < 1$	**converges** to 1.3247
$g(x) = x^3 - 1$	$5.26 > 1$	2.375, 12.40, 1904: diverges

A limit must satisfy $L = g(L)$; check it in the original equation.

Newton–Raphson follows tangents



- $f(x) = x^2 - 2$, drawn from $x_0 = 2$.
- Each tangent meets the axis at the next estimate.
- $2 \to 1.5 \to 1.4166667 \to 1.4142157$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Worked: Newton–Raphson on $x^3 - x - 1$

1. **Step 1** $x_1 = 1.5 - \frac{0.875}{5.75} = 1.347826$
2. **Step 2** $x_2 = 1.325200$
3. **Step 3** $x_3 = 1.324718$
4. **Check** $f(1.32465) < 0 < f(1.32475)$: 1.3247 to 4 d.p.

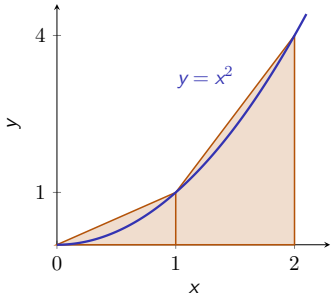
Decision

Three steps, against about fourteen for **bisection**; but it needs f' and a good start.

When Newton–Raphson fails

Situation	**What happens**
$f'(x_0) = 0$	division by zero: the tangent never meets the axis
$f'(x_0)$ close to 0	the next estimate jumps a long way
start near a different root	**converges** to the wrong root
tangent from $x_0 = \frac{1}{\sqrt{3}}$ on $x^3 - x - 1$	$f' = 0$: no next estimate

The trapezium rule, drawn



- $\int_0^2 x^2\,dx$, two strips, $h = 1$.
- $\frac{1}{2}[0 + 2(1) + 4] = 3$.
- Exact $\frac{8}{3} = 2.667$: an overestimate.

$$\int_a^b y\,dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right]$$

More strips, smaller error

Strips	h	Estimate	Error
2	1	3	0.333
4	0.5	$0.25[0 + 2(3.5) + 4] = 2.75$	0.083
8	0.25	2.6875	0.021

Exact value $\frac{8}{3} = 2.6667$.

Over or under?

Convex, $f'' > 0$

Chords lie above the curve.

Trapezium rule overestimates: x^2 on $[0, 2]$.

Concave, $f'' < 0$

Chords lie below the curve.

Trapezium rule underestimates: $\sqrt{x}$ on $[1, 4]$.

Check yourself

Question	**Answer**
Does $\tan x$ have a root in $(1, 2)$?	no: asymptote at $\frac{\pi}{2}$
Interval proving 2.718 to 3 d.p.?	$(2.7175, 2.7185)$
Iteration converges when...	$\lvert g' \rvert < 1$ near the root
Newton–Raphson fails when...	$f'(x_n) = 0$
Trapezium rule on a concave curve?	underestimate