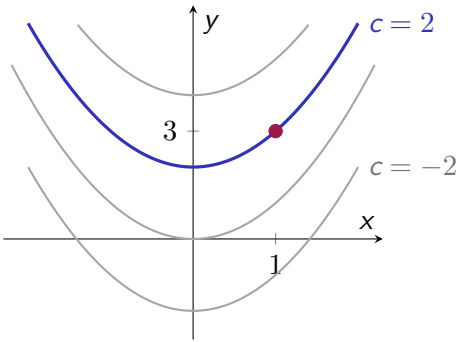
Integration

AQA A-level Mathematics 7357, section H: antiderivatives, area, substitution, parts, partial fractions, differential equations

Standard integrals

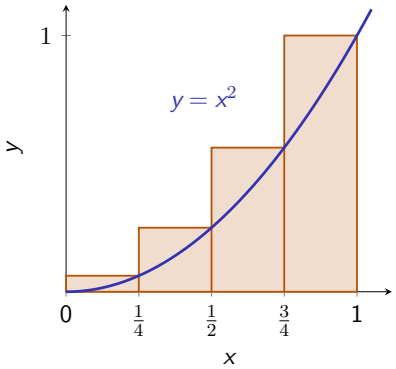
$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1} + c$	$\cos kx$	$\frac{1}{k}\sin kx + c$
$\frac{1}{x}$	$\ln \lvert x \rvert + c$	$\sin kx$	$-\frac{1}{k}\cos kx + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$	$\sec^2 kx$	$\frac{1}{k}\tan kx + c$

A family of antiderivatives



- $\frac{dy}{dx} = 2x \Rightarrow y = x^2 + c$.
- Same gradient at every x, shifted vertically.
- Through $(1, 3)$: $c = 2$.

Area as a limit of rectangles



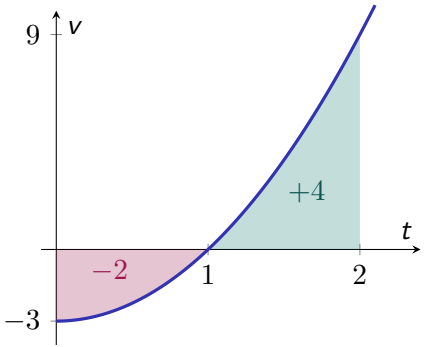
- $y = x^2$ on $[0, 1]$, 4 right-end strips.
- $\frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) = \frac{15}{32} = 0.469.$
- Exact area $\frac{1}{3}$: more strips close the gap.

The Fundamental Theorem

1. **Antiderivative** $F(x) = \frac{x^3}{3}$ for $f(x) = x^2$
2. **Evaluate** $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0$
3. **Result** $\frac{1}{3} \approx 0.333$, below the estimate 0.469
4. **Reverse** $\frac{d}{dx}\int_0^x t^2\,dt = x^2$

$$\int_a^b f(x)\,dx = F(b) - F(a), \qquad F' = f$$

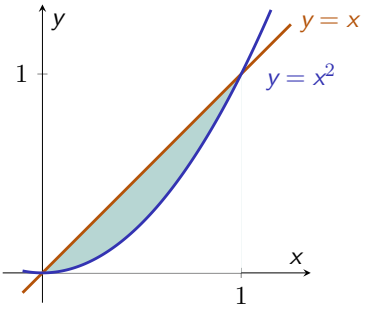
Displacement or distance?



- $v = 3t^2 - 3$, $0 \le t \le 2$; $v = 0$ at $t = 1$.
- Displacement $\int_0^2 v\,dt = -2 + 4 = 2$ m.
- Distance $|-2| + 4 = 6$ m.

Split the integral where the integrand changes sign, then add the sizes.

Area between two curves



- $y = x$ and $y = x^2$ meet at $x = 0$ and $x = 1$.
- Top minus bottom, then integrate.
- $\int_0^1 (x - x^2)\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

$$A = \int_a^b \left(y_{\text{top}} - y_{\text{bottom}}\right) dx$$

Integrating the standard forms

1. **Exponential** $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$
2. **Reciprocal** $\int \frac{2}{x}\,dx = 2\ln|x| + c$
3. **Trig** $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + c$
4. **Definite** $\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 0 - (-1) = 1$

Integration by substitution

1. **Choose** $\int_0^1 2x(x^2+1)^4\,dx,\ u=x^2+1,\ du=2x\,dx$
2. **Limits** $x=0 \Rightarrow u=1;\ x=1 \Rightarrow u=2$
3. **Integrate** $\int_1^2 u^4\,du = \left[\frac{u^5}{5}\right]_1^2$
4. **Evaluate** $\frac{32-1}{5} = \frac{31}{5}$

Decision

Keep the old limits only if you change back to x first.

Numerator = derivative of denominator

1. **Spot** $\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + c$
2. **Adjust** $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + c$
3. **Trig** $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Integration by parts

1. **Product** $u = x$, $\frac{dv}{dx} = e^x$: $\int xe^x\,dx = xe^x - \int e^x\,dx$
2. **Result** $= xe^x - e^x + c$
3. **Log** $u = \ln x$, $\frac{dv}{dx} = 1$: $\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx$
4. **Result** $= x\ln x - x + c$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Integrating with partial fractions

1. **Split** $\dfrac{5x+1}{(x-1)(x+2)} \equiv \dfrac{2}{x-1} + \dfrac{3}{x+2}$
2. **Integrate** $2\ln|x-1| + 3\ln|x+2| + c$
3. **Definite** $\int_2^3 \ldots \, dx = 2\ln 2 + 3\ln\frac{5}{4} = \ln\frac{125}{16}$

Decision

Combine logs only at the end, using the log laws.

Separable differential equations

1. **Model** $\frac{dP}{dt} = 0.1P$, $P = 200$ at $t = 0$
2. **Separate** $\int \frac{1}{P}\, dP = \int 0.1\, dt$
3. **Integrate** $\ln P = 0.1t + c$, so $P = Ae^{0.1t}$
4. **Condition** $A = 200$: $P = 200e^{0.1t}$

$$\frac{dy}{dx} = f(x)g(y) \;\Rightarrow\; \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

A model: Newton's law of cooling

1. **Given** room at 20 °C; $T = 90$ at $t = 0$, 55 at $t = 10$
2. **Solve** $\int \frac{dT}{T-20} = -\int k\,dt \Rightarrow T - 20 = Ae^{-kt}$
3. **Start** $t = 0$: $A = 70$
4. **Rate** $35 = 70e^{-10k}$: $k = \frac{\ln 2}{10} = 0.0693$
5. **Predict** $t = 30$: $T = 20 + 70e^{-2.08} = 28.75$ °C

Decision

The excess over room temperature halves every 10 minutes.

Traps in integration

Slips

$\int \frac{1}{x}\,dx = \frac{x^0}{0}$

Dropping $+c$, then being given a point.

Integrating across a sign change for a total area.

Habits

$\int \frac{1}{x}\,dx = \ln|x| + c$: the $n = -1$ case.

Keep $+c$; use the point to fix it.

Find the roots first, split, add the sizes.

Check yourself

Question	**Answer**
$\int 6x^2\,dx$?	$2x^3 + c$
$\int_1^e \frac{1}{x}\,dx$?	1
$\int e^{-2x}\,dx$?	$-\frac{1}{2}e^{-2x} + c$
Method for $\int x\cos x\,dx$?	by parts, $u = x$
$\frac{dy}{dx} = xy$: separated?	$\int \frac{dy}{y} = \int x\,dx$