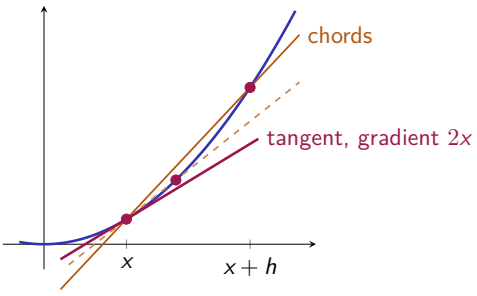
Differentiation

AQA A-level Mathematics 7357, section G: first principles, stationary points, the rules, implicit and parametric, rates, optimisation

Standard derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$\sin kx$	$k\cos kx$
e^{kx}	ke^{kx}	$\cos kx$	$-k\sin kx$
$\ln x$	$\frac{1}{x}$	$\tan kx$	$k\sec^2 kx$
a^x	$a^x \ln a$	constant	0

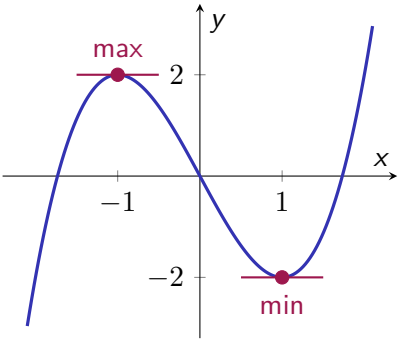
From secant to tangent



- Gradient of a chord, then let $h \to 0$.
- $\frac{(x+h)^2-x^2}{h} = 2x + h \to 2x$.

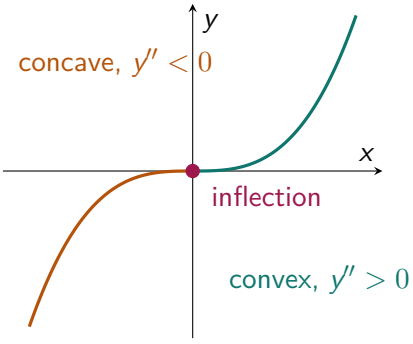
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Stationary points, drawn



x	$y'' = 6x$	**Nature**
-1	$-6 < 0$	maximum
1	$6 > 0$	minimum

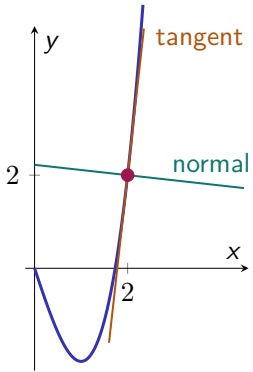
Concavity and inflection



- $f' > 0$: **convex**; $f' < 0$: **concave**.
- Inflection: f' changes sign.
- $y = x^3$: stationary at 0, yet keeps rising.

Trap: $y = x^4$ has $y''(0) = 0$ but is a minimum; check the sign change.

Tangent and normal



- $y = x^3 - 3x$ at $x = 2$: $y = 2$.
- $y'(2) = 9$: tangent $y = 9x - 16$.
- **Normal** gradient $-\frac{1}{9}$: $y - 2 = -\frac{1}{9}(x - 2)$.

The chain rule

1. **Power** $y = (3x^2 + 1)^5$: $\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot 6x = 30x(3x^2 + 1)^4$
2. **Exponential** $y = e^{\sin x}$: $\frac{dy}{dx} = \cos x\, e^{\sin x}$
3. **Log** $y = \ln(x^2 + 4)$: $\frac{dy}{dx} = \frac{2x}{x^2+4}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The product rule

1. **Split** $y = xe^x$: $u = x,\ v = e^x$
2. **Rule** $\frac{dy}{dx} = u'v + uv' = e^x + xe^x$
3. **Check** $x = 0$: $e^0 + 0 = 1$, so the curve rises through the origin
4. **Stationary** $e^x(1 + x) = 0$: $x = -1$

Decision

$(uv)' \neq u'v'$: at $x = 1$, $u'v' = e$ but the true gradient is $2e$.

$$\frac{d}{dx}(uv) = u'v + uv'$$

The quotient rule

1. **Parts** $u = \ln x$, $u' = \frac{1}{x}$; $v = x$, $v' = 1$
2. **Rule** $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{1 - \ln x}{x^2}$
3. **Stationary** $1 - \ln x = 0$: $x = e$, $y = \frac{1}{e}$
4. **Nature** gradient $+$ before e, $-$ after: a maximum

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Implicit differentiation

1. **Differentiate** $x^2 + y^2 = 25 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$
2. **Rearrange** $\frac{dy}{dx} = -\frac{x}{y}$
3. **At (3, 4)** $\frac{dy}{dx} = -\frac{3}{4}$
4. **Product term** $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$

Decision

The same $-\frac{3}{4}$ as the tangent found with circle geometry.

$$\frac{d}{dx}\left(y^n\right) = ny^{n-1}\frac{dy}{dx}$$

Parametric differentiation

1. **Given** $x = t^2$, $y = 2t$
2. **Rates** $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2$
3. **Divide** $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$
4. **At t = 2** point $(4, 4)$, gradient $\frac{1}{2}$
5. **Tangent** $y - 4 = \frac{1}{2}(x - 4)$, so $x - 2y + 4 = 0$

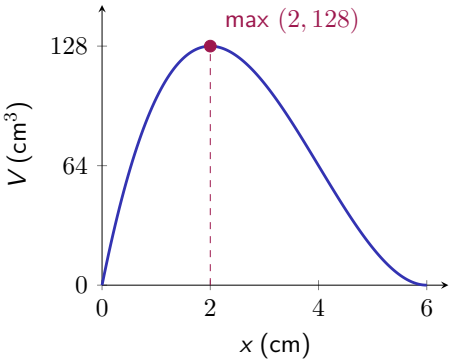
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Connected rates of change

1. **Known** $\frac{dr}{dt} = 0.2$ cm/s, $r = 5$ cm
2. **Formula** $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2 = 100\pi$
3. **Chain** $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 100\pi \times 0.2$
4. **Result** $20\pi \approx 62.8$ cm^3/s

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Optimisation: the biggest box



- Cut squares of side x from a 12×12 sheet.
- $V = x(12 - 2x)^2$,
  $V' = (12 - 2x)(12 - 6x)$.
- $x = 2$: $V = 128$; $x = 6$ gives $V = 0$.

Traps in differentiation

Slips

Calling every $f'(x) = 0$ a turning point.

Ignoring the endpoints of a restricted interval.

Differentiating $\sin x$ with x in degrees.

Habits

Classify with f'' or the sign change; x^3 is an inflection.

Compare stationary and boundary values for a global extreme.

Work in radians: $\frac{d}{dx}\sin x = \cos x$ only then.

Check yourself

Question	Answer
$\frac{d}{dx}\sqrt{x}$?	$\frac{1}{2\sqrt{x}}$
$\frac{d}{dx}\sin 3x$?	$3\cos 3x$
$\frac{d}{dx}x^2 \ln x$?	$2x \ln x + x$
$f'(a) = 0$, $f''(a) > 0$ means?	minimum
Gradient of $y^2 = x$ at $(4, 2)$?	$\frac{1}{2y} = \frac{1}{4}$