Exponentials and logarithms

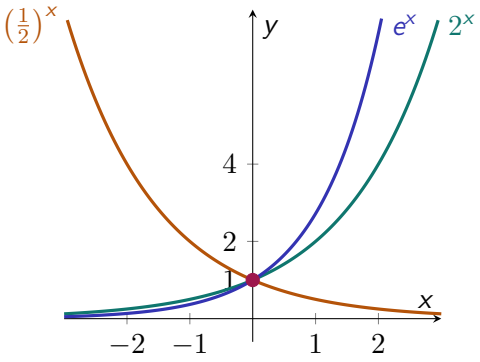
AQA A-level Mathematics 7357, section F: exponential graphs, e and ln, log laws, equations, models, linear reduction

Index laws become log laws

Index law	**Log law**
$a^m a^n = a^{m+n}$	$\log_a(uv) = \log_a u + \log_a v$
$a^m \div a^n = a^{m-n}$	$\log_a \frac{u}{v} = \log_a u - \log_a v$
$(a^m)^k = a^{mk}$	$\log_a(u^k) = k \log_a u$
$a^0 = 1$, $a^1 = a$	$\log_a 1 = 0$, $\log_a a = 1$

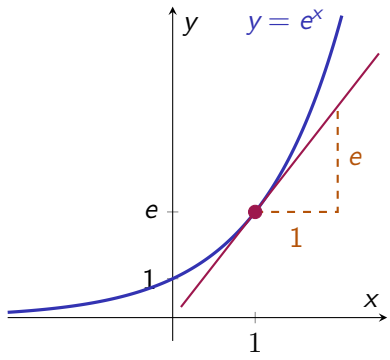
$\ln$ means $\log_e$; $\log_a y$ is defined only for $y > 0$.

Exponential graphs



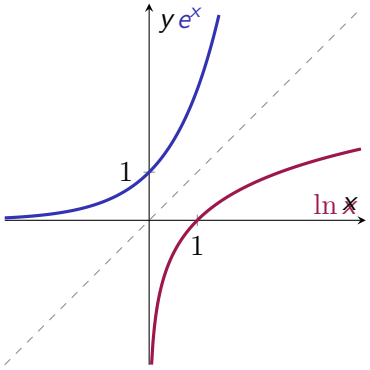
- All pass through $(0, 1)$.
- $y = 0$ is an **asymptote**.
- $a > 1$ grows; $0 < a < 1$ decays.

e^x is its own gradient



- At every point, gradient = height.
- At $(1, e)$ the tangent has gradient e.
- $\frac{d}{dx}e^{kx} = ke^{kx}$.

$\ln x$ undoes e^x



- Reflections in $y = x$.
- $\ln x$ is defined for $x > 0$ only.
- $e^{\ln x} = x$, $\ln(e^x) = x$.

Solving $a^x = b$

1. **Logs** $3^x = 20 \Rightarrow \ln(3^x) = \ln 20$
2. **Power law** $x \ln 3 = \ln 20$
3. **Divide** $x = \dfrac{\ln 20}{\ln 3} = 2.727$
4. **Check** $3^{2.727} = 20.0$ ✓

$$a^x = b \iff x = \frac{\ln b}{\ln a}$$

A log equation: check the domain

1. **Domain** $\log_2 x + \log_2(x-2) = 3$ needs $x > 2$
2. **Combine** $\log_2\big(x(x-2)\big) = 3$
3. **Remove log** $x(x-2) = 2^3 = 8$, so $x^2 - 2x - 8 = 0$
4. **Solve** $(x-4)(x+2) = 0$: $x = 4$ or $x = -2$

Decision

$x = 4$ only: $x = -2$ makes $\log_2(-2)$ undefined.

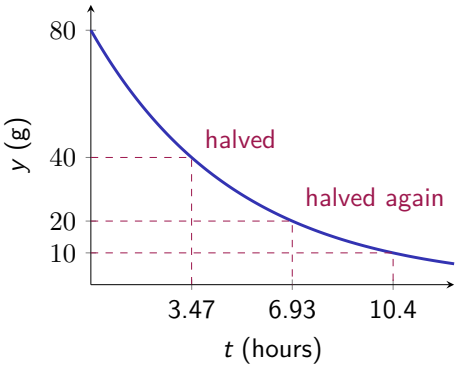
A hidden quadratic in 2^x

1. **Substitute** $2^{2x} - 6 \cdot 2^x + 8 = 0 \Rightarrow u^2 - 6u + 8 = 0$
2. **Factorise** $(u-2)(u-4) = 0$: $u = 2$ or $u = 4$
3. **Back** $2^x = 2$: $x = 1$; $2^x = 4$: $x = 2$

Decision

A root $u \leq 0$ would give no solution: 2^x is always positive.

Decay and half-life



- $y = 80e^{-0.2t}$, grams, hours.
- $40 = 80e^{-0.2t} \Rightarrow t = \frac{\ln 2}{0.2}$.
- **Half-life** 3.47 h, from any start.
- In general $t_{1/2} = \dfrac{\ln 2}{|k|}$.

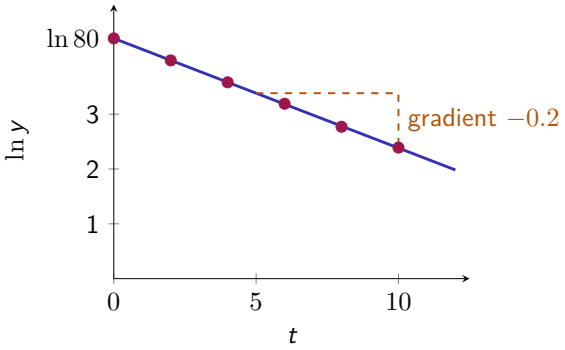
Fitting a growth model

1. **Given** $P = Ae^{kt}$; $P = 500$ at $t = 0$, 800 at $t = 5$
2. **A** $t = 0$: $A = 500$
3. **k** $800 = 500e^{5k}$, $e^{5k} = 1.6$, $k = \frac{\ln 1.6}{5} = 0.0940$
4. **Predict** $t = 10$: $P = 500e^{0.940} = 500 \times 1.6^2 = 1280$
5. **Judge** compare with a fresh measurement before trusting it

Decision

Extrapolating far beyond the data assumes the rate k never changes.

Straight lines from curves: $\ln y$ against t



- $y = Ae^{kt} \Rightarrow \ln y = \ln A + kt$.
- Gradient k, intercept $\ln A$.
- Here $A = 80$, $k = -0.2$.

Which plot straightens which model?

Model	Plot	Gradient	Intercept
$y = kb^x$	$\log y$ against x	$\log b$	$\log k$
$y = ax^n$	$\log y$ against $\log x$	n	$\log a$

Points $(\log x, \log y) = (0.301, 1.380)$ and $(0.903, 3.186)$: $n = \frac{1.806}{0.602} = 3$, $\log a = 0.477$, so $y = 3x^3$.

Traps with logs

Wrong

$\log(u+v) = \log u + \log v$

$\ln(2x) = 2\ln x$

Keeping a root that makes a log argument negative.

Right

$\log(uv) = \log u + \log v$; $\log(u+v)$ does not split.

$\ln(2x) = \ln 2 + \ln x$; $\ln(x^2) = 2\ln x$.

State the domain first and test every candidate.

Check yourself

Question	**Answer**
$\log_2 32$?	5
Solve $e^{2x} = 7$.	$x = \frac{1}{2} \ln 7$
$\frac{d}{dx} e^{3x}$?	$3e^{3x}$
Half-life of $Ae^{-0.05t}$?	$20 \ln 2 \approx 13.9$
$y = ax^n$: plot what?	$\log y$ against $\log x$